OPERATIONS RESEARCH

Dr. Soniya Dhama
Assistant Professor
Department of Mathematical Sciences
RGIPT, Jais

The Simplex Method

An Iterative Approach to Linear Programming Optimization

Dr. Soniya Dhama, RGIPT

The Simplex Algorithm: A Revolutionary Solution

Developed by *George Dantzig* in 1947, the Simplex Method provides an efficient algorithmic technique for solving linear programming problems of any magnitude-from two variables to thousands.

From Graphical to Algebraic Solutions

Find Initial Trial Solution	Identify an initial basic feasible solution to the problem
Test for Optimality	Determine whether the current solution is optimal
Improve and Iterate	If not optimal, apply systematic rules to improve non-optimal solutions until optimality is reached

Key Characteristics of the Simplex Method

1. Iterative Procedure

Systematically searches through feasible solutions, moving from one basic feasible solution to an improved neighboring solution

2. Maximization Focus

Concentrates on maximization problems, though minimization problems can be easily converted through standard transformations

3. **Focuses on Feasible Solutions**

Selects the optimal solution exclusively from the set of feasible solutions.

NOTE: We only address problems where the initial basic feasible solution is non-degenerate.

Standard Form of LPP

Consider a linear programming problem after introducing slack and surplus variables.

$$\text{Max. } Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n + 0x_{n+1} + 0x_{n+2} + \cdots + 0x_{n+m}$$

subject to

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1j}x_j + \cdots + a_{1n}x_n + x_{n+1} &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2j}x_j + \cdots + a_{2n}x_n + x_{n+2} &= b_2 \\
\ldots \qquad\qquad & \quad \ldots \\
\ldots \qquad\qquad & \quad \ldots \\
a_{\mathrm{m}1}x_1 + a_{\mathrm{m}2}x_2 + \cdots + a_{\mathrm{m}j}x_j + \cdots + a_{\mathrm{m}n}x_n + x_{n+\mathrm{m}} &= b_{\mathrm{m}}
\end{aligned}$$

$x_i \geq 0$ for all $i = 1,2,\ldots,N$, where $N = n + m$, and $b_1, b_2, \ldots, b_m$ are all positive.

Matrix Representation

$$\text{Max. } Z = \mathbf{cx}$$

subject to

$$A\mathbf{x} = \mathbf{b}, \qquad \mathbf{x} \geq \mathbf{0},$$

where

$A = [a_{ij}]_{m \times N}$ is the coefficient matrix of order $m \times N$,

$$\mathbf{x} = [x_1, x_2, x_3, \dots, x_n, \dots, x_N]_{N \times 1}$$
$$\mathbf{c} = (c_1, c_2, \dots, c_n, 0,0, \dots, 0)_{1 \times N}$$

and

$$\mathbf{b} = [b_1, b_2, \dots, b_m]_{m \times 1}$$

Standard Form Components

The linear programming problem in matrix form:

- **A**: Coefficient matrix (m×N)
- **x**: Decision variable vector
- **b**: Resource constraint vector
- **c**: Objective function coefficient vector

Column vectors denoted by [·] without transpose; row vectors by (·).

Objective Function in Basis Form

- For a given **basic variable vector**

$$\mathrm{x}_B = [x_{B1}, x_{B2}, \dots, x_{Bm}],$$

  the corresponding **cost vector** is

$$\mathbf{c}_B = (c_{B1}, c_{B2}, \dots, c_{Bm}).$$

- c_{Bi} are the **coefficients of the basic variables** in the objective function

- Since all **non-basic variables = 0** in a BFS

$$Z = c_{B1}x_{B1} + c_{B2}x_{B2} + \cdots + c_{Bm}x_{Bm} + 0$$

Or

$$Z = \mathbf{c}_B \mathrm{x}_B$$

Interpretation:

- Objective value depends **only on basic variables**
- Cost contribution from non-basic variables is **zero**

Fundamental Theorem of Linear Programming

If a linear programming problem

$$\max Z = cx \;\; subject\ to \;\; Ax = b, \;\; x \geq 0$$

has an optimal feasible solution, then at least one basic feasible solution must be optimal.

This powerful result justifies the Simplex Method's approach: instead of examining infinite feasible solutions, we need only investigate the finite set of basic feasible solutions.

From Feasible to Basic Feasible Solutions

Reduction Theorem

Any feasible solution of an LPP

$$\max Z = cx \ \ subject\ to \ \ Ax = b, x \geq 0$$

can be systematically reduced to a basic feasible solution through a constructive algorithm.

- The procedure identifies linear dependencies among column vectors and eliminates variables until linear independence is achieved.

Key Implication

If an optimal feasible solution exists, the reduction procedure preserves optimality, guaranteeing an optimal basic feasible solution.

Optimal Feasible Solution to Optimal BFS

Given: Optimal feasible solution: $x^* = (x_1, x_2, \dots, x_k, 0, \dots, 0), x_i > 0$

Associated columns: $\alpha_1, \alpha_2, \dots, \alpha_k$ *are linearly dependent*

Step 1
(Find Dependence Relation)

There exist scalars $\lambda_1, \dots, \lambda_k$,not all zero, such that $\sum_{i=1}^{k} \lambda_i \alpha_i = 0$

Choose signs so that **at least one** $\lambda_i > 0$

Step 2
(Define Scaling Factor)

$$v = \max_{1 \le i \le k} \left(\frac{\lambda_i}{x_i} \right)$$

Ensures $v > 0$

Step 3
(Construct New Solution)

$$x_i' = x_i - \frac{\lambda_i}{v}, \quad i = 1, \dots, k,$$

$$x_j' = 0, \qquad j = k+1, \dots, N$$

Step 4
(Feasibility Check)

From definition of v such that $x_i' \geq 0 \; \forall i$

So $x' \geq 0, \; Ax' = b$ implies

x' is feasible

Step 5
(Variable Reduction)

Since $v = \lambda_i / x_i$ for at least one i, $x_i' = 0$.

Number of non-zero variables **decreases by at least 1**

Step 6
(Iterate)

If remaining columns are **linearly independent**

→ BFS obtained

Else:

→ Repeat Steps 1–5

(Optimality is preserved)

Example

Consider the set of equations

$$5x_1 - 4x_2 + 3x_3 + x_4 = 3$$
$$2x_1 + x_2 + 5x_3 - 3x_4 = 0$$
$$x_1 + 6x_2 - 4x_3 + 2x_4 = 15$$
$$x_1, x_2, x_3, x_4 \geq 0$$

If $x_1 = 1, x_2 = 2, x_3 = 1, x_4 = 3$ is a feasible solution then find a basic feasible solution.

Given System

$$Ax = b$$

$$\begin{bmatrix} 5 & -4 & 3 & 1 \\ 2 & 1 & 5 & -3 \\ 1 & 6 & -4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 15 \end{bmatrix}$$

Column Representation

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 = b$$

where

$$\alpha_1 = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}, \ \alpha_2 = \begin{bmatrix} -4 \\ 1 \\ 6 \end{bmatrix}, \ \alpha_3 = \begin{bmatrix} 3 \\ 5 \\ -4 \end{bmatrix},$$

$$\alpha_4 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \ b = \begin{bmatrix} 3 \\ 0 \\ 15 \end{bmatrix}$$

Given Feasible Solution

Feasible Solution

$$x = (1,2,1,3)$$

Verification

$$\alpha_1 + 2\alpha_2 + \alpha_3 + 3\alpha_4 = b$$

Observation

- All variables are **non-zero**
- Number of variables $k = 4$
- Since $m = 3$, vectors must be **linearly dependent**

Linear Dependence Relation

Assume

$$\alpha_1 = a\alpha_2 + b\alpha_3 + c\alpha_4$$

Solve System

$$\begin{aligned} -4a + 3b + c &= 5 \\ a + 5b - 3c &= 2 \\ 6a - 4b + 2c &= 1 \end{aligned}$$

Solution

$$a = \frac{22}{43}, \qquad b = \frac{139}{86}, \qquad c = \frac{189}{86}$$

Construct λ-Relation

Multiply to Clear Denominators

$$86\alpha_1 - 44\alpha_2 - 139\alpha_3 - 189\alpha_4 = 0$$

Identify λ Values

$$\lambda_1 = 86,$$

$$\lambda_2 = -44,$$

$$\lambda_3 = -139,$$

$$\lambda_4 = -189$$

Compute Scaling Factor v

Given

$$x = (1,2,1,3)$$

Compute

$$v = \max\left(\frac{\lambda_1}{x_1}, \frac{\lambda_2}{x_2}, \frac{\lambda_3}{x_3}, \frac{\lambda_4}{x_4}\right)$$

$$v = \max(86, -22, -139, -63) = 86$$

Result

$$v = \frac{\lambda_1}{x_1} \Rightarrow x_1 \textit{ will be eliminated}$$

New Feasible Solution

Update Variables

$$x_i' = x_i - \frac{\lambda_i}{v}$$

Compute

$$x_1' = 0, \qquad x_2' = \frac{216}{86},$$

$$x_3' = \frac{225}{86}, \qquad x_4' = \frac{447}{86}$$

New Solution

$$x' = \left(0, \frac{216}{86}, \frac{225}{86}, \frac{447}{86}\right)$$

Verify BFS Condition

Basis Matrix

$$B = (\alpha_2, \alpha_3, \alpha_4)$$

Determinant

$$\begin{vmatrix} -4 & 3 & 1 \\ 1 & 5 & -3 \\ 6 & -4 & 2 \end{vmatrix} \neq 0$$

Conclusion

The new feasible solution is a **Basic Feasible Solution (BFS).**

Alternative BFS

Use Opposite Sign

$$-86\alpha_1 + 44\alpha_2 + 139\alpha_3 + 189\alpha_4 = 0$$

Resulting BFS

$$x_1 = \frac{225}{139}, \; x_2 = \frac{234}{139},$$

$$x_3 = 0, \; x_4 = \frac{228}{139}$$

Improved Basic Feasible Solution

Theorem (Improved BFS Condition)

If a non-basic column has positive reduced cost and at least one positive pivot coefficient, then pivoting on that column produces a new BFS with an improved (or equal) objective value; strictly improved in the non-degenerate case.

Let

$$\mathrm{x}_B = B^{-1}b$$

be a BFS with

$$Z = c_B \mathrm{x}_B$$

If there exists a column $\alpha_j \in A \setminus B$ such that

- **Reduced cost:**

$$c_j - Z_j > 0$$

- **Pivot condition:**

$$y_{ij} > 0 \; \textit{for at least one } i$$

Then

- A **new BFS** can be obtained by replacing one column of B with α_j ,and the new objective value satisfies

$$Z' \geq Z$$

- If the BFS is **non-degenerate**, then

$$Z' > Z$$

Procedure to Get Improved BFS

Objective: Find a **new BFS** from a given BFS that gives a **better (or equal) value of the objective function**

I. Representation Using Basis

Matrix Setup

$$A = (\alpha_1, \alpha_2, \dots, \alpha_N), B = (\beta_1, \beta_2, \beta_m)$$

Column Expansion

Each non-basic column can be written as:

$$\alpha_j = \sum_{i=1}^{m} y_{ij}\, \beta_i$$

or

$$Y_j = B^{-1}\alpha_j$$

Interpretation

- y_{ij} =**Transformed constraint coefficient**
- Shows how column $\alpha_j$is expressed in the current basis

II. Pivot & New BFS Construction

Choose Leaving Variable

Find index r such that:

$$v = \min_i \left\{ \frac{x_{Bi}}{y_{ij}} \middle| y_{ij} > 0 \right\}$$

Then: **Replace** β_r in B with α_j

Update Basic Variables

$$x'_{Bi} = x_{Bi} - \frac{y_{ij}}{y_{rj}} x_{Br}, i \neq r, \qquad x'_{Br} = \frac{x_{Br}}{y_{rj}}$$

Feasibility

- If $y_{rj} > 0 \rightarrow x'_B \geq 0$
- New solution is a **BFS**

Improvement in Objective Value

Objective Function Update

$$Z' = Z + v(c_j - Z_j)$$

Conclusions

If $c_j - Z_j > 0$ and $v > 0$ → **Strict improvement:**

$$Z' > Z$$

If $v = 0$ → **Degenerate BFS:**

$$Z' = Z$$

New Basis

$$B' = (\beta_1, \ldots, \alpha_j, \ldots, \beta_m)$$

- If the initial BFS is **degenerate**, the new BFS is also **degenerate**

Example

Max $Z = 3x_1 + 2x_2$

Subject to

$$\begin{aligned} x_1 + x_2 &\le 4 \\ 2x_1 + x_2 &\le 5 \\ x_1, x_2 &\ge 0 \end{aligned}$$

Standard Form

$$\begin{aligned} 1x_1 + x_2 + s_1 &= 4 \\ 2x_1 + x_2 + s_2 &= 5 \end{aligned}$$

Initial BFS:

$x_1 = 0,\ x_2 = 0,\ s_1 = 4,\ s_2 = 5$

Initial $Z = 0$

Check Improvement

- **Reduced costs:**

  $c_1 - Z_1 = 3 > 0$, $c_2 - Z_2 = 2 > 0.$

  → **Enter:** x_1

- **Pivot column (** x_1**:)**

  Coefficients = $1, 2$, both > 0

  → **Pivot possible**

- **Minimum Ratio Test**

  $\frac{4}{1} = 4, \quad \frac{5}{2} = 2.5 \Rightarrow$ *Leave:* s_2

- **New BFS (After Pivot)**

  $x_1 = 2.5, x_2 = 0, s_1 = 1.5, s_2 = 0$

- **Improved Objective**

  $Z' = 3(2.5) = 7.5 > 0$

What Can Happen in a Linear Programming Problem?

Understanding all the possibilities is crucial for solving linear programming problems effectively.

At any iteration of the **Simplex Method**, a **Basic Feasible Solution (BFS) can lead to several distinct outcomes**.

Four Possible Outcomes :

1. Optimal Solution

The best possible solution that maximizes or minimizes the objective function while satisfying all constraints.

2. Alternative Optimal Solutions

Multiple different solutions that all achieve the same optimal objective function value.

3. Unbounded Solution

The objective function can increase or decrease without limit, indicating no finite optimal value exists.

4. Consistency or Redundancy in Constraints

Constraints that either contradict each other or contain unnecessary equations.

Simplex Method: Computational Procedure

First Step

- **Convert to Maximization**

If minimization problem, multiply objective function by -1.

- **Ensure Non-Negative RHS**

Make all $b_i \geq 0$ by multiplying constraints by -1 if needed.

- **Convert to Equations**

Add slack/surplus/artificial variables as required.

Building the Initial Simplex Table

- **Find Initial B.F.S.**

Use methods from previous cases. Apply two-phase or Big M-method if artificial variables present.

- **Construct Table**

Set up initial simplex table with basis, variables, and solution values. Non-basic variables always equal zero.

- **Calculate Net Evaluations**

Compute $\Delta_j = c_j - c_B Y_j$ for each variable. In starting table, these equal c_j values and are zero for basic variables.

Initial Simplex Table

B	c_B	c_j	c_1	c_2	…	c_k	…	c_n	c_{n+1}	c_{n+2}	…	c_{n+m}	Min Ratio x_B/Y_k
		x_B	$Y_1\ (=\alpha_1)$	$Y_2\ (=\alpha_2)$		$Y_k\ (=\alpha_k)$		$Y_n\ (=\alpha_n)$	$Y_{n+1}\ (\beta_1)$	$Y_{n+2}\ (\beta_2)$		$Y_{n+m}\ (\beta_m)$	
Y_{n+1}	$c_{B1}=0$	$x_{B1}=b_1$	$y_{11}=a_{11}$	$y_{12}=a_{21}$	…	$y_{1k}=a_{1k}$	…	$y_{1n}=a_{1n}$	1	0	…	0	
Y_{n+2}	$c_{B2}=0$	$x_{B2}=b_2$	$y_{21}=a_{21}$	$y_{22}=a_{22}$	…	$y_{2k}=a_{2k}$	…	$y_{2n}=a_{2n}$	0	1	…	0	
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
Y_{n+r}	$c_{Br}=0$	$x_{Br}=b_r$	$y_{r1}=a_{r1}$	$y_{r2}=a_{r2}$	…	**$y_{rk}=a_{rk}$**	…	$y_{rn}=a_{rn}$	0	0	…	0	→
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
Y_{n+m}	$c_{Bm}=0$	$x_{Bm}=b_m$	$y_{m1}=a_{m1}$	$y_{m2}=a_{m2}$	…	$y_{mk}=a_{mk}$	…	$y_{mn}=a_{mn}$	0	0	…	1	
$Z=\mathbf{c}_B\mathbf{x}_B=0$		Δ_j	Δ_1	Δ_2		Δ_k ↑		Δ_n	Δ_{n+1}	Δ_{n+2}		Δ_{n+m}	

Optimality Condition Theorem

If for every column which is in A but not in basis B, the reduced cost is non-positive (for maximization), then the current basic feasible solution is optimal.

- This fundamental theorem provides the stopping criterion for the Simplex Method. When all reduced costs for non-basic variables are non-positive, no improvement is possible, confirming optimality.

Given

Current BFS:

$$x_B = B^{-1}b$$

Objective value:

$$Z^* = c_B x_B$$

Condition

If for all $\alpha_j \notin B$:

$$c_j - Z_j \leq 0$$

Conclusion

x_B is an **Optimal Basic Feasible Solution**
Z^* is the **Optimal Value**

Alternative Optimal Solutions

(More than one solution gives the **same optimal value.)**

A linear programming problem possesses alternative optimal solutions when multiple distinct variable sets achieve the same optimal objective function value.

Non-Basic Alternative

If reduced cost equals zero for a non-basic variable and all its pivot column coefficients are non-positive, a non-basic alternative optimal solution exists.

Basic Alternative

If reduced cost equals zero for a non-basic variable and at least one pivot column coefficient is positive, an alternative basic optimal solution exists.

Non-Basic Alternative Optimum

If:

$$c_j - Z_j = 0$$

and

$$y_{ij} \leq 0, \forall i$$

Result

A non-basic alternative optimal solution exists

Basic Alternative Optimum

If:

$$c_j - Z_j = 0$$

and

$$y_{ij} > 0 \text{ for at least one } i$$

Result

A new optimal BFS exists (after pivot)

Optimality Testing Criteria

- **Optimal Solution**

If $\Delta_j = c_j - c_B Y_j \leq 0$ for all j, solution is optimal. Alternative optimal solutions exist if any $\Delta_j = 0$ for non-basic variables.

- **Not Optimal**

If any $\Delta_j > 0$, **solution can be improved**. Proceed to iteration steps to find better solution.

- **Unbounded Solution**

If maximum positive Δ_j has all non-positive column elements, problem is unbounded.

- **No Feasible Solution**

If optimality satisfied but artificial variable remains in basis with non-zero value, no feasible solution exists.

Iteration Process: Improving the Solution

1. Select Entering Vector

Choose variable with **largest positive** Δ_j value to enter basis (incoming vector).

2. Select departing Vector

Calculate **minimum ratio** x_{Bi}/y_{ik} for positive y_{ik}. Corresponding variable **leaves basis** (outgoing vector).

3. Identify Pivot Element

Element at intersection of incoming column and outgoing row. Transform to unity by row operations.

4. Update Simples Table

Perform row operations to make **pivot** column a **unit** vector. Replace outgoing with incoming variable in basis.

5. Test for Optimality

Repeat process until optimal solution achieved or special case identified.

Finding Initial Basic Feasible Solutions in LPP

A systematic method for determining the most convenient initial basic feasible solution to linear programming problems across different constraint types.

LPP:

Max/Min $Z = cx$ $s.t.$ $\mathrm{Ax} = b, \mathrm{x} \geq 0$

Why Do We Need a Starting BFS?

Simplex Requirement

Need a **basis matrix** B that is:

- **Square (m × m)**
- **Non-singular**
- Ideally an **identity matrix**

Goal

Choose m variables whose columns form I_m so we can compute:

$$\mathrm{x}_B = B^{-1}b = b \geq 0$$

Dr. Soniya Dhama, RGIPT

Three Cases of Constraint Types

Case I: ≤ Constraints

(All original constraints have ≤ sign.)

Method Overview

- Insert slack variables to convert inequalities into equations.
- The requirement vector must satisfy $b_i \geq 0$ (multiply constraint by -1 if negative).
- Initial basis matrix B consists of slack variable columns, forming an identity matrix.
- This yields immediate BFS.

Solution Process

- Set all non-basic (given) variables to zero
- Solve equations for slack variables
- Initial B.F.S. equals the b vector

Case II: ≥ Constraints

(All constraints have ≥ sign.)

Add Surplus Variables

- Convert constraints to equations by inserting surplus variables to account for excess.

Introduce Artificial Variables

- Surplus variables create negative identity, making initial basis infeasible.
- Add artificial variables to form proper basis.

Form Initial Basis

- Basis matrix B consists of artificial variable columns.
- Set non-basic variables to zero and solve for artificial variables.

Case III: Mixed Signs

(Constraints contain ≤, ≥, and = signs.)

- When constraints contain ≤, ≥, and = signs simultaneously, combine all variable types strategically.
- Use slack, surplus, and artificial variables.
- Basis matrix includes unit column vectors from slack and artificial variables.
- Initial B.F.S. obtained by setting non-basic variables to zero.

Slack Variables : For ≤ constraints

Surplus + Artificial : For ≥ constraints

Artificial Only : For = constraints

NOTE: In all three cases, initial BFS consists of constants $b_1, b_2, \ldots, b_m \geq 0$.

Example

" $\leq$ " Constraint **(Works Naturally)**

$$x_1 + x_2 \leq 5$$

Add slack variable:

$$x_1 + x_2 + s_1 = 5$$

Column of s_1 =[1,0,0, . . .]

- This **is an identity column**
- s_1 can be a **basic variable**

" = " Constraint **(No Identity Column)**

$$x_1 + x_2 = 5$$

- There is **no slack or surplus variable** to add
- Coefficients of x_1, x_2 are not identity-like.
- So, There is **no obvious basic variable** to start with

To Fix: Add an artificial variable:

$$x_1 + x_2 + a_1 = 5$$

Now a_1 forms an **identity column**

" $\geq$ " Constraint **(Still No Identity Column)**

$$x_1 + x_2 \geq 5$$

Subtract Surplus

$$x_1 + x_2 - t_1 = 5$$

Column of t_1 =[−1,0,0, . . .]

- This is **not an identity column**
- So t_1 **cannot be basic**

To Fix: Add artificial variable:

$$x_1 + x_2 - t_1 + a_1 = 5$$

Column of a_1 =[1,0,0, . . .].

- Now we have a **starting basis**

Artificial Variables — What & Why

What is an Artificial Variable?

A **temporary variable** added to a constraint to create an **initial Basic Feasible Solution** when one does not naturally exist.

Why Are They Needed?

In the Simplex Method, we must start with:

- A **basis matrix** B (identity columns)
- A **Basic Feasible Solution** $x_B \geq 0$

Some constraints **do not provide an identity column** after adding slack/surplus variables:

- **"=" constraints**
- **"≥" constraints** (even after subtracting a surplus variable)

When Do We Add Them?

Constraint Type	After Conversion	Artificial Variable
$\leq$	Add slack	❌ Not needed
$=$	No identity column	✅ Needed
$\geq$	Subtract surplus	✅ Needed

NOTE: Artificial variables **must be removed** before the final solution.

Unbounded Solutions Theorem

Positive reduced cost + non-positive pivot column ⇒ Unbounded (Maximization).

If for any basic feasible solution there exists a column in A but not in basis B where the **reduced cost is positive** and all **pivot column coefficients** are non-positive, then the problem has an unbounded solution when maximizing.

Mathematical Implication

When this condition holds, the objective function value can be increased arbitrarily by giving sufficiently large values to the entering variable.

Given

LPP:

$$\text{Max } Z = cx \; s.t. \; Ax = b, \; x \geq 0$$

Current BFS:

$$x_B = B^{-1}b$$

Condition

If there exists a column $\alpha_j \notin B$ such that

$$c_j - Z_j > 0$$

and

$$y_{ij} \leq 0 \;\; for\ all\ i = 1,2, \ldots, m$$

Conclusion

The LPP has an **unbounded solution**

Inconsistency and Redundancy

Redundancy in Constraints

A constraint system is **redundant** if:

$$r(A) = r(A \mid b) = k \leq n < m$$

Meaning

- System has **more constraint equations than needed**
- Some constraints **do not affect the feasible region**
- Number of redundant constraints:

$$m - k$$

$r(A) = \text{Rank}(A)$

Inconsistency in Constraints

The system:

$$Ax = b$$

is **inconsistent** if:

$$r(A) \neq r(A \mid b)$$

Meaning

No solution satisfies all constraints → **No feasible region**

Artificial Variables: Decision Rules

Case I: No Artificial Vectors

If basis B contains no artificial vectors and optimality is satisfied, the current solution is a valid **basic feasible solution**.

Case II: Zero-Level Artificial Vectors

If artificial vectors appear in the basis at zero level and optimality is satisfied, the system is consistent. If all coefficients are zero for a row with an artificial vector, that **constraint is redundant**.

Case III: Positive-Level Artificial Vectors

If at least one artificial vector appears in the basis at a positive level when optimality is satisfied, **no feasible solution exists** for the problem.

Case I — Feasible & Valid Solution

Conditions:
1. No artificial variables in basis
2. Optimality condition satisfied

Conclusion
Current solution is a **BFS**

Case II — Consistent & Redundant Solution

Conditions:
1. Artificial variables in basis
2. Their values = 0
3. Optimality condition satisfied

Conclusion
System is consistent
If corresponding $y_{ij} = 0$,then that constraint is redundant

Case III — Infeasible Solution

Condition
1. Artificial variable in basis
2. Value > 0
3. Optimality condition satisfied

Conclusion
No feasible solution exists

Summary Table

Use this 3-question rule.

1. **Can Z improve?**
→ Check $c_j - Z_j$

2. **Can I pivot?**
→ Check $y_{ij} > 0$

3. **What does it mean?**
Optimal / Unbounded / Alternative / Infeasible

Condition	Result
$c_j - Z_j \leq 0\ \forall j$	Optimal BFS
$c_j - Z_j > 0, all\ y_{ij} \leq 0$	Unbounded
$c_j - Z_j = 0, all\ y_{ij} \leq 0$	Non-basic Alternative Optimum
$c_j - Z_j = 0, some\ y_{ij} > 0$	Basic Alternative Optimum
Artificial var > 0 in basis	Infeasible
Artificial var = 0 in basis	Consistent / Possibly Redundant