OPERATIONS RESEARCH

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Duality in Linear Programming

A fundamental discovery in optimization theory that reveals the deep connection between primal and dual problems.

Primal: the original problem
Dual: the relate problem

The Core Concept

Some times, it's easier to solve the dual problem than the primal, making duality a powerful computational tool.

Two Problems, One Solution

Every linear programming problem has a corresponding dual problem. They are mathematical replicas of each other.

Opposite Directions

If the primal is a maximization problem, the dual will be minimization, and vice versa.

Linked Solutions

Knowing the optimal solution to one problem allows us to easily find the optimal solution to the other.

The Diet Problem

Two foods provide vitamins A and B. We need to meet minimum daily requirements at minimum cost.

Vitamin	Food		Minimum daily requirement
	A_1	A_2	
A	6	9	60
B	4	13	108
Cost (per unit)	Rs 12	Rs 18	

Formulating the Primal

Let x_1 and x_2 be the number of units of foods A_1 and A_2 to be purchased respectively.

Objective Function:

Minimize $Z_x = 12x_1 + 18x_2$

Constraints:

$$6x_1 + 9x_2 \geq 60,$$
$$4x_1 + 13x_2 \geq 108,$$
$$x_1, x_2 \geq 0.$$

The Dual Perspective

Wholesale Dealer's Problem

- A dealer sells vitamins A and B. He must set maximum per-unit prices such that the resulting prices of foods A_1 and A_2 don't exceed their market values.
- The foods have market value only because of their vitamin content.

Formulating the Dual

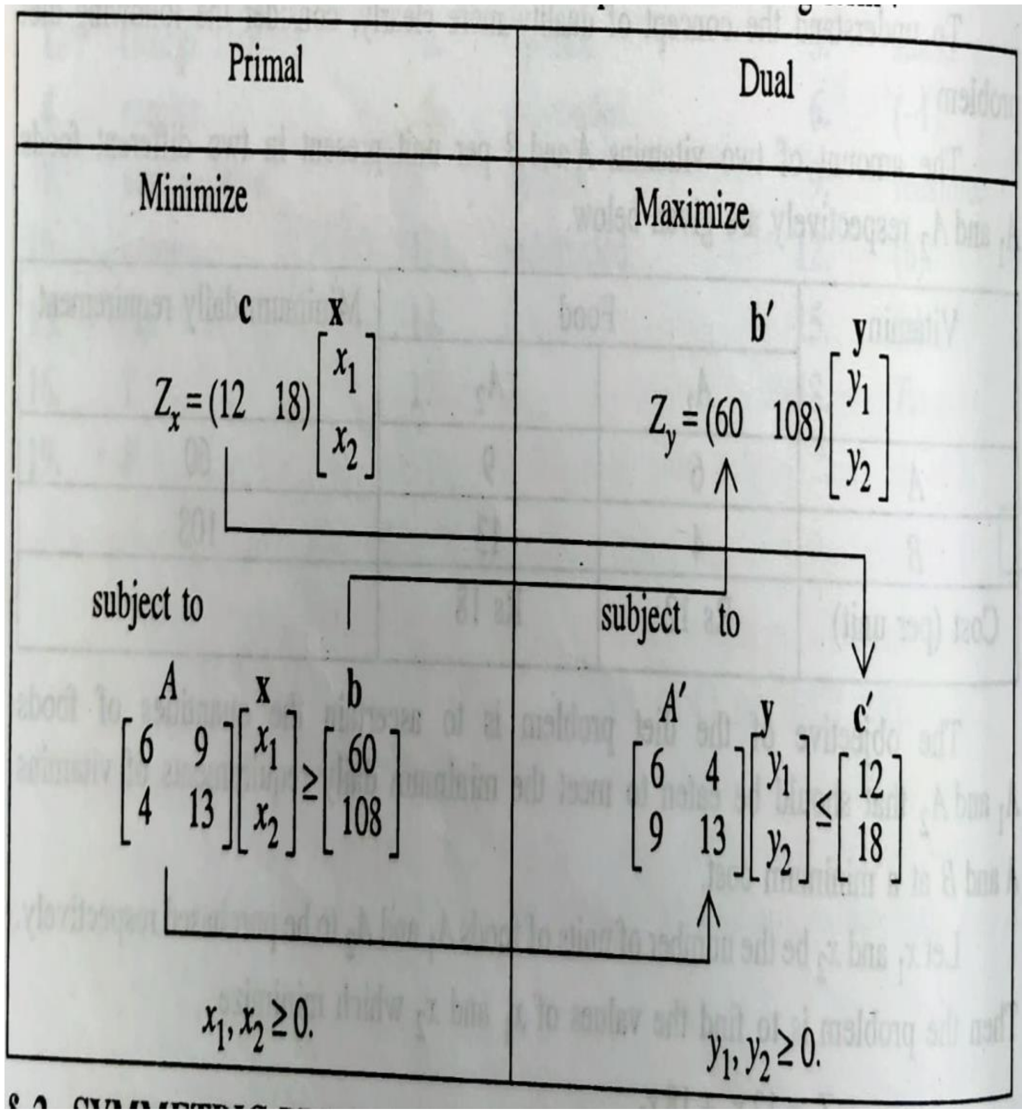
Variables: Let y_1 and y_2 are prices per unit of vitamins A and B, respectively.

Objective Function:

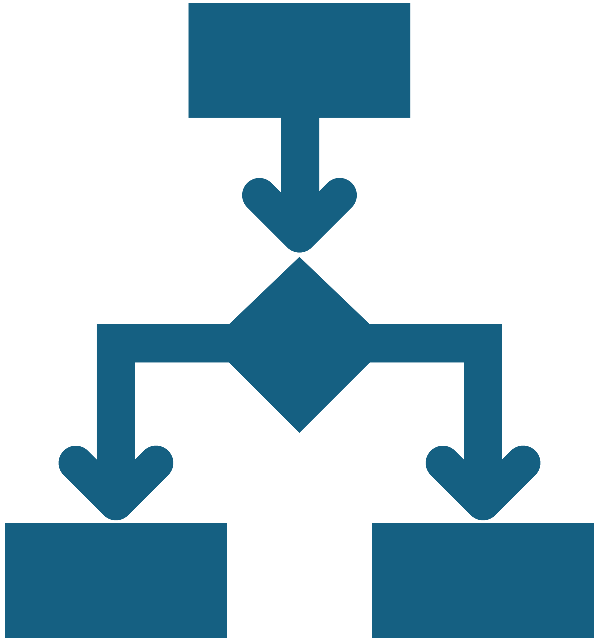
$$\text{Maximize } Z_y = 60y_1 + 108y_2$$

Constraints:

$$6y_1 + 4y_2 \leq 12$$
$$9y_1 + 13y_2 \leq 18,$$
$$y_1, y_2 \geq 0.$$



Primal-Dual Relationship



- **Problem Direction**

  Primal minimizes, dual maximizes (or vice versa).
- **Value Swap**

  Constraint values become objective coefficients and vice versa.
- **Matrix Transpose**

  Dual coefficient matrix is transpose of primal coefficient matrix.
- **Inequality Reversal**

  Direction of inequalities reverses between primal and dual.

Symmetric Primal-Dual Problems

Primal L.P.P.

Find the variables $x_1, x_2, \dots, x_n$ which **maximize**

$$Z_x = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

subject to

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \le b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \le b_2$$

$$\dots$$

$$\dots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \le b_m$$

and

$$x_1, x_2, \dots, x_n \ge 0$$

the signs of the parameters a, b, c are arbitrary.

Dual problem

Find the variables $w_1, w_2, \dots, w_m$ which **minimize**

$$Z_w = b_1w_1 + b_2w_2 + \dots + b_mw_m$$

subject to

$$a_{11}w_1 + a_{21}w_2 + \dots + a_{m1}w_m \ge c_1$$
$$a_{12}w_1 + a_{22}w_2 + \dots + a_{m2}w_m \ge c_2$$

$$\dots$$

$$\dots$$

$$a_{1n}w_1 + a_{2n}w_2 + \dots + a_{mn}w_m \ge c_n$$

and

$$w_1, w_2, \dots, w_n \ge 0$$

Matrix Form of Symmetric Primal-Dual Problem

Primal Problem.

Find a column vector $\mathbf{x} \in R^n$ which maximizes

$$Z_x = \mathbf{cx}, \mathbf{c} \in R^n$$

subject to

$$A\mathbf{x} \le \mathbf{b}, \mathbf{b} \in R^m,$$
$$x \ge 0 \text{ and } A \text{ is an } m \times n \text{ real matrix.}$$

Dual Problem.

Find a column vector $\mathrm{w} \in R^m$ which minimizes

$$Z_w = \mathbf{b}'\mathbf{w}$$

subject to

$$A'\mathbf{w} \ge \mathbf{c}'$$

$\mathbf{w} \ge 0, A', \mathbf{b}', \mathbf{c}'$ are the transposes of $A, \mathbf{b}$ and $\mathbf{c}$ respectively.

Example

Consider the symmetric primal problem

$$\text{Max. } Z_x = 5x_1 + 9x_2$$

subject to

$$\begin{aligned} x_1 &\le 6 \\ x_1 + x_2 &\le 13 \\ x_2 &\le 8 \\ x_1, x_2 &\ge 0 \end{aligned}$$

The corresponding dual problem is

$$\text{Min. } Z_w = 6w_1 + 13w_2 + 8w_3$$

subject to

$$\begin{aligned} w_1 + w_2 &\ge 5 \\ w_2 + w_3 &\ge 9 \\ w_1, w_2, w_3 &\ge 0 \end{aligned}$$

Unsymmetric Primal-Dual Problems

<u>Primal Problem</u>.

Find a column vector $\mathbf{x} \in R^n$ which maximizes

$$Z_x = \mathbf{cx}, \mathbf{c} \in R^n$$

subject to $A\mathbf{x} = \mathbf{b}, \mathbf{b} \in R^m$,
$\mathbf{x} \geq 0$ and A is an $m \times n$ real matrix.

<u>Dual Problem</u>.

Find a column vector $\mathrm{w} \in R^m$ which minimizes

$$Z_w = \mathbf{b}'\mathbf{w}$$

subject to $A'\mathbf{w} \geq \mathbf{c}'$.
In this case the dual variables are unrestricted in sign.

Key Takeaway: The dual variables corresponding to primal equality constraints must be unrestricted in sign and those associated with primal inequalities must be non-negative.

Dual of an LPP with Mixed Restrictions

1. Equation Handling:

Replace each equation in the primal problem with two inequalities going in opposite directions (≤ and ≥).

Example: the equation $2x_1 + 5x_2 = 9$ is replaced by

$$2x_1 + 5x_2 \leq 9 \text{ and } 2x_1 + 5x_2 \geq 9.$$

2. Sign Adjustment:

For maximization problems, ensure all constraints have ≤ sign, and for minimization problems, all constraints have ≥ sign. (Multiply by -1 if needed).

3. Unrestricted Variables:

Replace unrestricted variables with difference of two non-negative variables.

4. Find the Dual:

Now find the dual problem as usual procedure.

Standard Primal Form: An LPP is said to be in standard primal form if

(a) for a maximization problem all the constraints have ≤ sign.

(b) for a minimization problem all the constraints have ≥ sign.

Example 1.

Find the dual of the following LPP

$$\text{Min. } Z = 10x_1 + 20x_2$$

subject to

$$3x_1 + 2x_2 \geq 18$$
$$x_1 + 3x_2 \geq 8$$
$$2x_1 - x_2 \leq 6,$$
$$x_1, x_2 \geq 0$$

Standard primal form of given LPP:

$$\text{Min. } Z = 10x_1 + 20x_2$$

subject to

$$\begin{aligned} 3x_1 + 2x_2 &\geq 18 \\ x_1 + 3x_2 &\geq 8 \\ -2x_1 + x_2 &\geq -6 \\ x_1, x_2 &\geq 0 \end{aligned}$$

The matrix form of the primal problem is

Min. $Z = 10x_1 + 20x_2 = (10,20)[x_1, x_2] = \mathbf{cx}$

s.t.

$$\begin{bmatrix} 3 & 2 \\ 1 & 3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq \begin{bmatrix} 18 \\ 8 \\ -6 \end{bmatrix}$$

$$A\mathbf{x} \geq \mathbf{b}, x_1, x_2 \geq 0.$$

The dual of this problem is

Max. $Z_D = \mathbf{b}'\mathbf{y} = (18,8,-6)[y_1, y_2, y_3]$

s.t.

$$A'\mathbf{y} \leq \mathbf{c}' \text{ or } \begin{bmatrix} 3 & 1 & -2 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \leq \begin{bmatrix} 10 \\ 20 \end{bmatrix}$$

$$\text{or} \quad \begin{bmatrix} 3y_1 + y_2 - 2y_3 \\ 2y_1 + 3y_2 + y_3 \end{bmatrix} \leq \begin{bmatrix} 10 \\ 20 \end{bmatrix}$$

Hence dual of the given problem is

$$\text{Max. } Z_D = 18y_1 + 8y_2 - 6y_3$$

s.t.

$$3y_1 + y_2 - 2y_3 \leq 10,$$
$$2y_1 + 3y_2 + y_3 \leq 20,$$
$$y_1, y_2, y_3 \geq 0.$$

Example 2.

Write the dual of the following problem:

$$\text{Min. } Z = 2x_2 + 5x_3$$

subject to

$$\begin{aligned} x_1 + x_2 \quad\quad\quad &\geq 2 \\ 2x_1 + x_2 + 6x_3 &\leq 6 \\ x_1 - x_2 + 3x_3 &= 4 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

First, we convert the given problem to standard primal form:

$$\text{Min. } Z = 0x_1 + 2x_2 + 5x_3$$

subject to

$$\begin{aligned} x_1 + x_2 \quad\quad\quad &\geq 2 \\ -2x_1 - x_2 - 6x_3 &\geq -6 \\ x_1 - x_2 + 3x_3 &\geq 4 \\ -x_1 + x_2 - 3x_3 &\geq -4 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

The matrix form of the standard primal form is

$$\text{Min. } Z = (0,2,5)[x_1, x_2, x_3] = \mathbf{c}.\mathbf{x}$$

subject to

$$\begin{bmatrix} 1 & 1 & 0 \\ -2 & -1 & -6 \\ 1 & -1 & 3 \\ -1 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \geq \begin{bmatrix} 2 \\ -6 \\ 4 \\ -4 \end{bmatrix} \text{ or } A\mathbf{x} \geq \mathbf{b}, x_1, x_2, x_3 \geq 0$$

Thus, the dual of the given primal is

$$\text{Max. } Z_D = \mathbf{b}'.\mathbf{y} \quad = (2, -6, 4, -4)[y_1, y_2, y_3', y_3''] = 2y_1 - 6y_2 + 4(y_3' - y_3'')$$

subject to

$$A'y \leq c' \text{ or } \begin{bmatrix} 1 & -2 & 1 & 1 \\ 1 & -1 & -1 & -1 \\ 0 & -6 & 3 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3' \\ y_3'' \end{bmatrix} \leq \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}$$

$$\text{or } \begin{bmatrix} y_1 - 2y_2 + y_3' - y_3'' \\ y_1 - y_2 - y_3' + y_3'' \\ 0.y_1 - 6y_2 + 3y_3' - 3y_3'' \end{bmatrix} \leq \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}$$

$$y_1, y_2, y_3', y_3'' \geq 0$$

We can further write

$$\text{Max. } Z_D = 2y_1 - 6y_2 - 4(y_3' - y_3'')$$

subject to

$$y_1 - 2y_2 + (y_3' - y_3'') \leq 0$$
$$y_1 - y_2 - (y_3' - y_3'') \leq 2$$
$$-6y_2 + 3(y_3' - y_3'') \leq 5,$$
$$y_1, y_2, y_3', y_3'' \geq 0.$$

Substituting $y_3 = y_3' - y_3''$, **the required dual is**

$$\text{Max. } Z_D = 2y_1 - 6y_2 + 4y_3$$

subject to

$y_1 - 2y_2 + y_3 \leq 0, y_1 - y_2 - y_3 \leq 2, -6y_2 + 3y_3 \leq 5,$

$y_1, y_2 \geq 0$ and y_3 is unrestricted in sign.

Example 3

Write the dual of the following problem:

$$\text{Max. } Z = 3x_1 + 5x_2 + 7x_3$$

subject to

$$x_1 + x_2 + 3x_3 \leq 10$$

$$4x_1 - x_2 + 2x_3 \geq 15$$

$$x_1, x_2 \geq 0, x_3 \text{ is unrestricted.}$$

Substituting $x_3 = x_3' - x_3''$ we get the standard primal form of the given problem as

$$\text{Max. } Z = 3x_1 + 5x_2 + 7(x_3' - x_3'')$$

subject to

$$\begin{aligned} x_1 + x_2 + 3x_3' - 3x_3'' &\leq 10 \\ -4x_1 + x_2 - 2x_3 + 2x_3'' &\leq -15 \\ x_1, x_2, x_3', x_3'' &\geq 0. \end{aligned}$$

The matrix form of the above problem is:

subject to

$$\begin{bmatrix} 1 & 1 & 3 & -3 \\ -4 & 1 & -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3' \\ x_3'' \end{bmatrix} \leq \begin{bmatrix} 10 \\ -15 \end{bmatrix}$$

$$\text{or } A\mathbf{x} \leq \mathbf{b},\ x_1, x_2, x_3', x_3'' \geq 0.$$

The dual of the given problem is

$$\text{Min. } Z_D = \mathbf{b}'\mathbf{y} = (10, -15)[y_1, y_2] = 10y_1 - 15y_2$$

subject to $A\mathbf{y} \geq \mathbf{c}'$

$$\text{or } \begin{bmatrix} 1 & -4 \\ 1 & 1 \\ 3 & -2 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \geq \begin{bmatrix} 3 \\ 5 \\ 7 \\ -7 \end{bmatrix}$$

$$\text{or } \begin{bmatrix} y_1 - 4y_2 \\ y_1 + y_2 \\ 3y_1 - 2y_2 \\ -3y_1 + 2y_2 \end{bmatrix} \geq \begin{bmatrix} 3 \\ 5 \\ 7 \\ -7 \end{bmatrix}$$

Which is

$$\text{Min. } Z_D = 10y_1 - 15y_2$$

subject to

$$y_1 - 4y_2 \geq 3, \qquad y_1 + y_2 \geq 5,$$

$$3y_1 - 2y_2 \geq 7, \qquad -3y_1 + 2y_2 \geq -7$$

$$y_1, y_2 \geq 0.$$

Hence the required dual problem is

$$\text{Min. } Z_D = 10y_1 - 15y_2$$

subject to

$$y_1 - 4y_2 \geq 3,$$
$$y_1 + y_2 \geq 5,$$
$$3y_1 - 2y_2 = 7,$$
$$y_1, y_2 \geq 0.$$

DUALITY RESULTS

- The dual of a dual of the given primal is the primal itself.

- If $\mathbf{x}$ is any feasible solution to the primal problem

$$Max\ Z_p = \mathbf{cx}, \qquad \text{s.t.} \qquad A\mathbf{x} \le \mathbf{b}, \qquad \mathbf{x} \ge \mathbf{0}$$

  and $\mathbf{w}$ is any feasible solution to the dual problem

$$Min\ Z_D = \mathbf{b}'\mathbf{w} \qquad \text{s.t.} \qquad A'\mathbf{w} \ge \mathbf{c}', \qquad \mathbf{w} \ge \mathbf{0},$$

  then $\mathbf{cx} \le \mathbf{b}'\mathbf{w}$, *that is*, $Z_P \le Z_D$.

- The necessary and sufficient condition for any linear programming problem and its dual to have optimal solution is that both have feasible solutions.

- (Basic Duality theorem.) If $\mathbf{x_0}$ is an optimum solution to the primal, then there exists a feasible solution w_0 to the dual such that

$$c\mathbf{x_0} = \mathbf{b}' w_\mathbf{0}$$

  where $\mathbf{b}'$ is the transpose of $\mathbf{b}$.

- If any of the constraints in the primal is a perfect equality, the corresponding dual variable is unrestricted in sign.

- If any variable of the primal is unrestricted in sign, the corresponding constraint in the dual will be a strict equality.

Fundamental Duality Theorem

1) If either the primal or the dual problem has a finite optimal solution then the other problem also has a finite optimal solution and the optimal values of the objective function in both the problems are the same.

2) If primal (dual) problem has an unbounded optimum solution, the other problem has either no solution at all or an unbounded solution.

Existence Theorems

1. There exists a bounded (finite) optimum solution to an LPP if and only if there exists a feasible solution to both primal and its dual.

2. If there does not exist any feasible solution to the dual (primal) but there exists at least one to the primal (dual), then there does not exist any finite optimum solution to the primal (dual).

3. If there does not exist any finite optimum solution to the primal (dual) then there does not exist any feasible solution to the dual (primal).

	Primal Problem	Dual Problem
1.	Objective function Max. Z_P.	Objective function Min. Z_D.
2	Requirement vector.	Price vector.
3	Coefficient matrix A	Transpose of the coefficient matrix, A' or A^T
4	Constraints with $\leq$ sign.	Constraints with $\geq$ sign.
5	Relation.	Variable.
6	i-th inequality	i-th variable $w_i \geq 0$.
7	i-th constraint an equality.	i-th variable w_i unrestricted in sign.
8	Variable.	Relation.
9	i-th variable $x_i > 0$.	i-th relation a strict inequality.
10	i-th variable x_i unrestricted in sign.	i-th constraint a strict equality.
11	i-th slack variable positive.	i-th variable zero.
12	i-th variable zero.	i-th surplus variable positive.
13	Finite optimal solution.	Finite optimal solution with equal optimal value of objective function.
14	Unbounded solution	No solution or an unbounded solution

Relationship Between Final Simplex Tables of Primal and Dual

From the **final simplex table of the primal problem**, we can directly read the **optimal solution of the dual problem**, and **vice-versa**.

RULES:

- The optimal value of the primal objective function is equal to the optimal value of the dual objective function.

$$\boxed{\max Z_P = \min Z_D}$$

- In the final simplex table of the primal problem, take the values of

$$\Delta_j = c_j - Z_j$$

and change the sign for the slack (or surplus) variables.

The resulting numbers give the optimal values of the corresponding dual variables in the final simplex table of the dual problem.

- If either problem has unbounded solution, then the other will have no feasible solutions.

Why Duality Matters

Computational Advantage:

Sometimes it is easier to solve the dual than the primal problem.

- Compare the primal and dual problems.
- Identify which one has fewer constraints.
- First, solve that problem using the simplex method.
- Then, use the final simplex table to read the solution of the other problem using the duality rules.

Applications:

- ➢ Physics

  Used in:
  - Parallel circuit theory
  - Series circuit theory

- ➢ Economics

  Applied in:
  - Input–output models
  - Resource allocation systems

- ➢ Game Theory

  Used to find optimal strategies. For example:

  If Player B minimizes losses, duality allows us to Convert Player A's problem into Player B's problem and vice-versa

Example

Write the dual of the following problem and solve it.

$$\text{Max.}\, Z = 4x_1 + 2x_2$$

subject to

$$-x_1 - x_2 \leq -3$$
$$-x_1 + x_2 \leq -2,$$
$$x_1, x_2 \geq 0$$

Hence or otherwise write down the solution of the primal.

The given problem is in standard primal form.
Thus, the dual to the given primal is

$$\text{Min. } Z_D = -3w_1 - 2w_2$$

subject to

$$-w_1 - w_2 \geq 4$$
$$-w_1 + w_2 \geq 2$$
$$w_1, w_2 \geq 0.$$

Changing the objective function to maximization and introducing surplus variables: $w_3 \geq 0, w_4 \geq 0$ and artificial variables $w_{a_1}, w_{a_2} \geq 0$ the above dual problem reduces to

$$\text{Max. } Z_D' = 3w_1 + 2w_2 + 0w_3 + 0w_4 - Mw_{a_1} - Mw_{a_2}$$

subject to

$$-w_1 - w_2 - w_3 + w_{a_1} = 4$$
$$-w_1 + w_2 - w_4 + w_{a_2} = 2$$

Taking $w_1 = 0 = w_2 = w_3 = w_4$, we get $w_{a_1} = 4, w_{a_2} = 2$.

Now applying the simplex method to obtain the optimal solution, we have

	c_j	**3**	**2**	**0**	**0**			
B $\mathbf{c}_B$	$\mathbf{w}_B$	W_1	W_2	W_3	W_4	W_{a_1}	W_{a_2}	Min. ratio
W_{a_1} $-M$	4	-1	-1	-1	0	1	0	Negative
W_{a_2} $-M$	2	-1	[1]	0	-1	0	1	2 (min) →
$Z'_D = -6M$	Δ_j	$3 - 2M$	2 ↑	$-M$	- M	0	0 ↓	
W_{a_1} $-M$	6	-2	0	-1	-1	1	1	
W_2 **2**	2	-1	1	0	-1	0	1	
$Z'_D = 4 - 6M$	Δ_j	$5 - 2M$	0	$-M$	$2 - M$	0	-2	

Since no $\Delta_j > 0$ and a non-zero artificial variable appears in the basis therefore the dual problem does not possess any optimum basic feasible solution.

Consequently, the given problem does not possess any finite optimal solution.

Example

Write the dual of the following linear programming problem and hence solve it.

$$\text{Max. } Z = 3x_1 - 2x_2$$

subject to

$$x_1 \leq 4$$
$$x_2 \leq 6$$
$$x_1 + x_2 \leq 5$$
$$-x_2 \leq -1$$
$$x_1, x_2 \geq 0.$$

The given problem is in standard primal form.

The dual of the given primal is

$$\text{Min. } Z_D = 4w_1 + 6w_2 + 5w_3 - w_4$$

subject to

$$w_1 + w_3 \geq 3$$
$$w_2 + w_3 - w_4 \geq -2$$
$$w_1, w_2, w_3, w_4 \geq 0$$

Changing the dual problem to maximization and introducing surplus variable w_5 and slack variable w_6 to change the inequalities into equations, the dual problem becomes:

$$\text{Max. } Z'_D = -4w_1 - 6w_2 - 5w_3 + w_4 + 0w_5 + 0w_6$$

subject to

$$w_1 + w_3 \quad -w_5 = 3$$
$$-w_2 - w_3 + w_4 \quad +w_6 = 2$$

$$w_1, w_2, \ldots, w_6 \geq 0.$$

Taking $w_2 = 0, w_3 = 0$, we get $w_1 = 3, w_6 = 2$ which is the starting B.F.S.

The solution by simplex method is given in the following table :

	c_j	-4	-6	-5	1	0	0	Min. ratio
B c_B	w_B	W_1	W_2	W_3	W_4	W_5	W_6	w_B/W_4
W_1 -4	3	1	0	1	0	-1	0	Inf.
W_6 0	2	0	-1	-1	[1]	0	1	2 (min) →
$Z_D' = -12$	Δ_j	0	-6	-1	1 ↑	-4	0	
W_1 -4	3	1	0	1	0	-1	0	
W_4 1	2	0	-1	-1	1	0	1	
$Z_D' = -10$	Δ_j	0	-5	0	0	-4	-1	

The optimal solution of the dual is

$$w_1 = 3, w_2 = 0, w_3 = 0, w_4 = 2$$
$$\text{Min. } Z_D = -\text{ Max. } Z_D' = 10$$

The optimal solution of the primal problem is

$$x_1 = -\Delta_5 = 4, x_2 = -\Delta_6 = 1 \text{ and Max. } Z = \text{ Min. } Z_D = 10.$$

Sensitivity Analysis in LPP

Understanding how changes in model parameters affect optimal solutions

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What is Sensitivity Analysis?

Definition

- In linear programming, model parameters can change within certain limits without altering the optimum solution. This concept is called **sensitivity analysis**.
- Sensitivity analysis examines how changes in LP parameters, such as resource availability or profit coefficients, affect the optimal solution without resolving the entire problem.

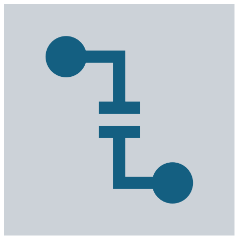
Why Managers Need It

Real-world parameters are uncertain. Sensitivity analysis helps managers understand how robust their optimal solution is to changes in costs, resources, or market conditions.

Sensitivity analysis studies how changes affect the current optimal solution.

Post-optimal analysis finds a new optimal solution after parameter changes.

Two Types of Parameter Changes



Right-Hand Side Changes : Alter resource availability (machine hours, raw materials, budget).

These changes shift constraint boundaries parallel to their original position.

1. Capacity expansion decisions
2. Budget allocation
3. Resource availability

Objective Coefficient Changes : Modify profit or cost coefficients.

These changes alter the slope of the objective function, potentially changing which solution is optimal.

1. Price fluctuations
2. Cost variations
3. Revenue projections

Both types provide crucial economic insights for decision-making under uncertainty.

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Graphical Sensitivity Analysis

Case 1: Sensitivity to the changes in the resource availability.

Case 2: Sensitivity to the changes in coefficients of the objective function.

Case 1: Changes in Resource Availability

Consider the following example:

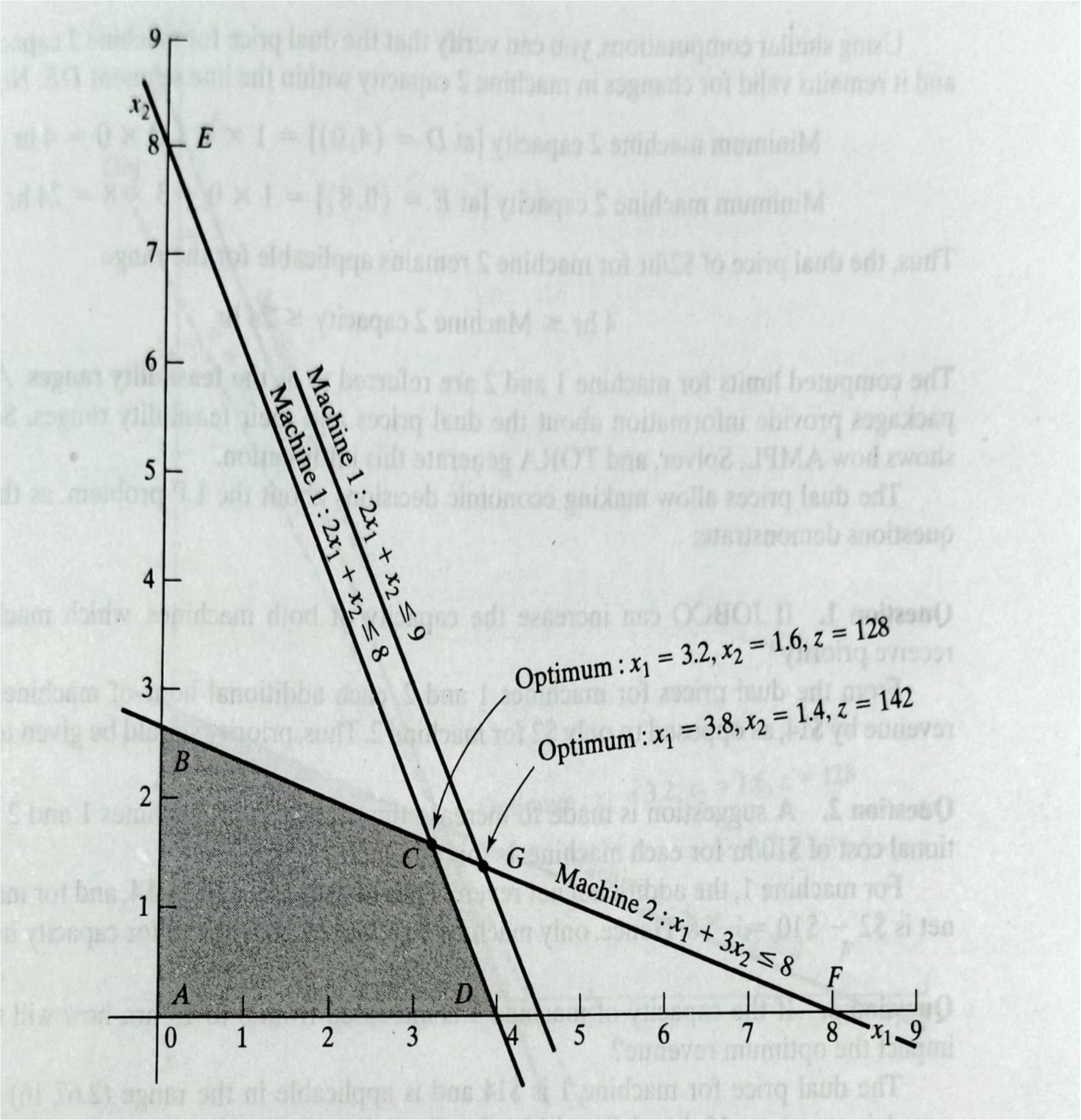
Sharma Industries manufactures two products on two machines. A unit of product 1 requires 2 hrs on machine 1 and 1 hr on machine 2. For product 2, one unit requires 1 hr on machine 1 and 3 hrs on machine 2 . The revenues per unit of products 1 and 2 are \$30 and \$20, respectively. The total daily processing time available for each machine is 8 hrs.

Letting x_1 and x_2 represent the daily number of units of products 1 and 2, respectively, the LP model is given as

$$\text{Maximize } z = 30x_1 + 20x_2$$

subject to

$$2x_1 + x_2 \leq 8 \text{ (Machine 1)}$$
$$x_1 + 3x_2 \leq 8 \text{ (Machine 2)}$$
$$x_1, x_2 \geq 0.$$



Graphical sensitivity of optimal solution to changes in the availability of resources (right-hand side of the constraints)

- This figure illustrates the change in the optimum solution when changes are made in the capacity of machine 1.
- If the daily capacity is increased from 8 to 9 hrs, the new optimum will move to point G.
- The rate of change in optimum z resulting from changing machine 1 capacity from 8 to 9 hrs can be computed as:

$$\begin{pmatrix} \text{Rate of revenue change} \\ \text{resulting from increasing} \\ \text{machine 1 capacity by 1hr} \\ \text{(point C to point G)} \end{pmatrix} = \frac{z_G - z_C}{(\text{ Capacity change })}$$

$$= \frac{142 - 128}{9 - 8} = \$14/\text{hr}$$

- The computed rate provides a direct link between the model input (resources) and its output (total revenue).
- It says that a unit increase (decrease) in machine 1 capacity will increase (decrease) revenue by $14.

Understanding Dual (Shadow) Prices

The Concept

- When machine 1 capacity increases from 8 to 9 hours, revenue increases by $14. This rate is called the **dual price.**

- The dual price can be interpreted as "**unit worth of resource**", i.e., it represents the change in the objective function value per unit increase in a resource's availability.

- **Graphical Interpretation:** The dual price equals the slope of the objective function at the optimal point relative to the constraint.

- **Machine 1**

Dual price: **$14 per hour**

- **Machine 2**

Dual price: **$2 per hour**

The dual price tells us which resource to prioritize for capacity expansion.

- In Figure, we can see that the dual price of $\$14/\mathrm{hr}$ remains valid for changes (increases or decreases) in machine 1 capacity that move its constraint parallel to itself to any point on the line segment BF.

- We compute machine 1 capacities at points B and F as follows:

$$\text{Minimum machine 1 capacity [at } B = (0{,}2.67)] = 2 \times 0 + 1 \times 2.67 = 2.67\text{hr}$$

$$\text{Minimum machine 1 capacity [at } F = (8{,}0)] = 2 \times 8 + 1 \times 0 = 16\text{hr}$$

- The conclusion is that the dual price of $\$14.00/\mathrm{hr}$ remains valid only in the range

$$\mathbf{2.67hr \leq \ Machine\ 1\ capacity\ \leq 16hr}$$

- Similarly, the dual price for machine 2 capacity is $\$2/\mathrm{hr}$, which remains valid for changes in machine 2 capacity within the line segment DE.

- Machine 2 capacities at points D and E are as follows:

$$\text{Minimum machine 2 capacity [at } D = (4{,}0)] = 1 \times 4 + 3 \times 0 = 4\text{hr}$$

$$\text{Minimum machine 2 capacity [at } E = (0{,}8)] = 1 \times 0 + 3 \times 8 = 24\text{hr}$$

- Thus, the dual price of $\$2/\mathrm{hr}$ for machine 2 remains applicable for the range

$$\mathbf{4hr \leq \ Machine\ 2\ capacity\ \leq 24hr}$$

The computed limits for machine 1 and 2 are referred to as the feasibility ranges. Changes outside this range produce a different dual price (worth per unit).

Feasibility Ranges and Managerial Questions

What is a Feasibility Range?

The range of right-hand side values for which the current optimal basis remain feasible. Outside this range, different constraints become binding, changing the dual price.

Which Machine to Expand?

Calculate dual price of both machines. The machine with the higher dual price should be expanded, as it provides more value per additional unit of capacity.

Is Capacity Expansion Profitable?

Compare the dual price to the cost of expansion. If dual price exceeds expansion cost, invest.

For example, if machine 1's dual price is Rs 120/hour and expansion costs Rs 100/hour, expand.

What Happens Outside the range?

When resource availability exceeds the feasibility range, the optimal solution changes structure, i.e., the same constraints no longer determine the solution, requiring re-optimization.

Question 1. If Sharma Industries can increase the capacity of both machines, which machine should receive priority?

Answer: From the dual prices for machines 1 and 2, each additional hour of machine 1 increases revenue by $14, as opposed to only $2 for machine 2. Thus, priority should be given to machine 1.

Question 2. A suggestion is made to increase the capacities of machines 1 and 2 at the additional cost of $10/hr for each machine. Is this advisable?

Answer: For machine 1, the additional net revenue per hour is $14 - 10 = \$4$, and for machine 2 , the net is $\$2 - \$10 = -\$8$. Hence, only machine 1 should be considered for capacity increase.

Question 3. If the capacity of machine 1 is increased from 8 to 13 hrs, how will this increase impact the optimum revenue?

Answer: The dual price for machine 1 is $14 and is applicable in the range $(2.67 \text{ hr}, 16 \text{ hr})$. The proposed increase to 13 hrs falls within the feasibility range. Hence, the increase in revenue is $\$14(13 - 8) = \$70$, which means that the total revenue will be increased from $128 to $198 $(= \$128 + \$70)$.

Question 4. Suppose that the capacity of machine 1 is increased to 20 hrs , how will this increase affect the optimum revenue?

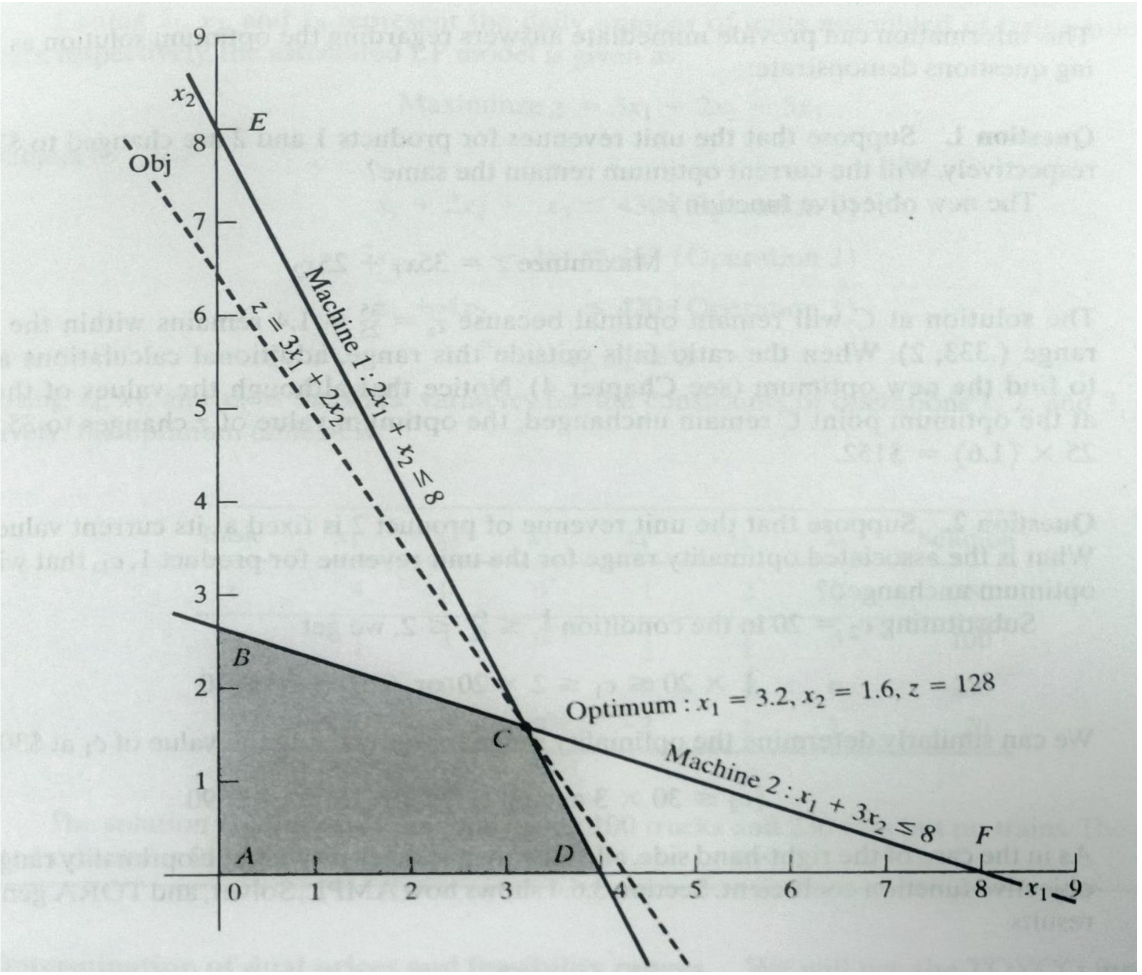
Answer: The proposed change is outside the feasibility range $(2.67 \text{ hr}, 16 \text{ hr})$. Thus, we can only make an immediate conclusion regarding an increase up to 16 hrs . Beyond that, further calculations are needed to find the answer .

Question 5. How can we determine the new optimum values of the variables associated with a change in a resource?

Answer: The optimum values of the variables will change.

Note: Falling outside the feasibility range does not mean that the problem has no solution. It only means that available information is not sufficient to make a complete decision.

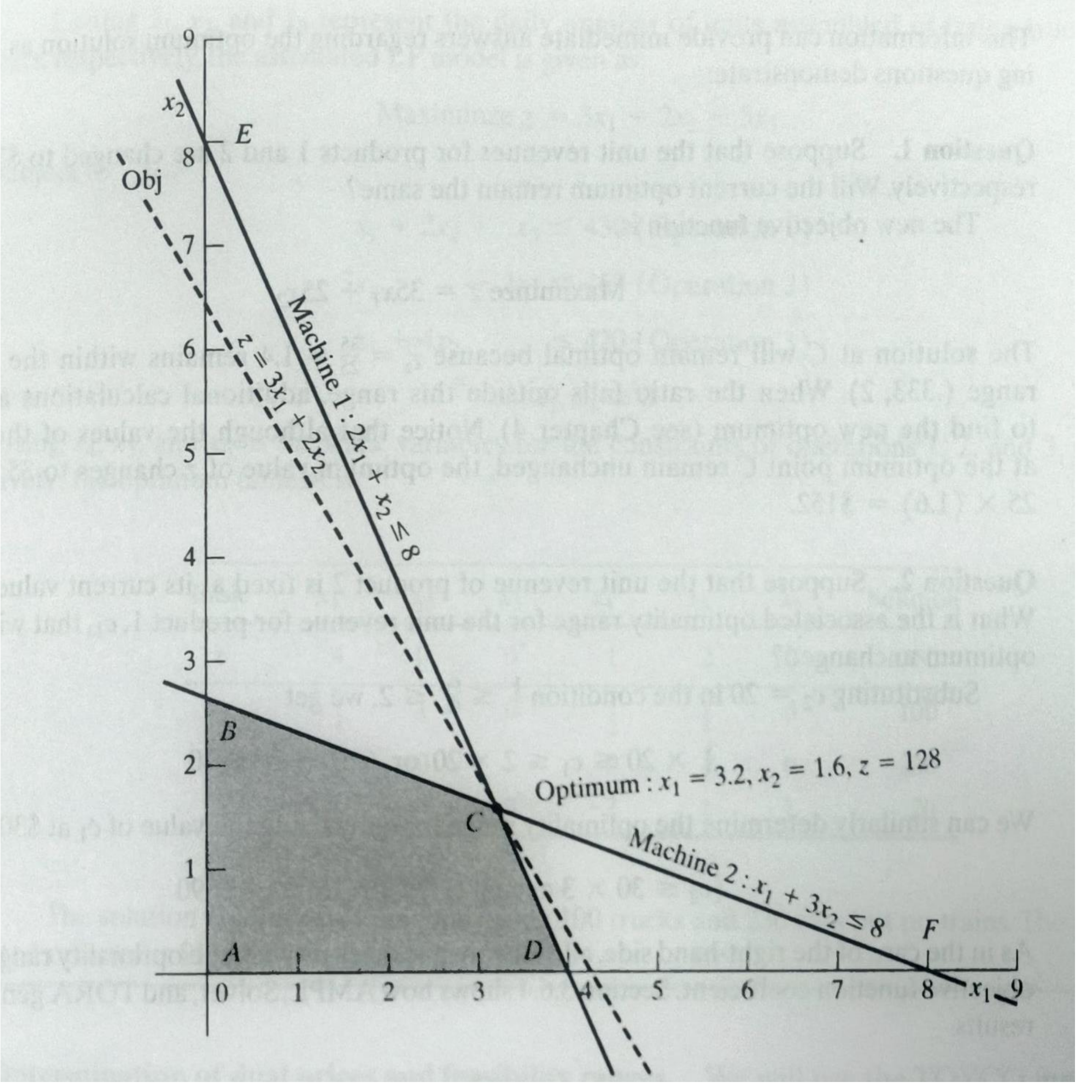
Case 2: Changes in Objective Coefficients



Graphical sensitivity of optimal solution to changes in the revenue units (coefficients of the objective function)

- This figure shows the graphical solution space of the Sharma Industries problem. The optimum occurs at point C$(x_1 = 3.2, x_2 = 1.6, z = 128)$.
- Changes in revenue units (i.e., objective-function coefficients) will change the slope of the objective function line.
- However, the current optimum remains optimal as long as these changes keep the solution within a specific range.
- As can be seen, the optimum solution at point C remains unchanged so long as the objective function lies between lines BF and DE.

Ranges for revenue units that will keep the optimum solution unchanged



- Consider the general objective function:

$$Max. z = c_1 x_1 + c_2 x_2$$

- At optimal point C, the objective function line z can rotate while still touching point C.

- The optimal solution remains unchanged as long as the objective line lies between the two binding constraints:

$$x_1 + 3x_2 = 8 \quad \text{and} \quad 2x_1 + x_2 = 8$$

- Comparing slopes:

$$\frac{-c_1}{c_2} \text{ must lie between } \frac{-1}{3} \text{ and } -2$$

- Hence, the ratio of coefficients must satisfy:

$$\frac{1}{3} \le \frac{c_1}{c_2} \le 2 \quad \text{or} \quad 0.333 \le \frac{c_1}{c_2} \le 2$$

- Within this range, point C remains the optimal solution.

Question 1. Suppose that the unit revenues for products 1 and 2 are changed to \$35 and \$25, respectively. Will the current optimum remain the same?

Answer: The new objective function is

$$\text{Maximinze } z = 35x_1 + 25x_2$$

The solution at C will remain optimal because $\frac{c_1}{c_2} = \frac{35}{25} = 1.4$ remains within the optimality range $(.333, 2)$. When the ratio falls outside this range, additional calculations are needed to find the new optimum.

Note that although the values of the variables at the optimum point C remain unchanged, the optimum value of z changes to $35 \times (3.2) + 25 \times (1.6) = \$152$.

Question 2. Suppose that the unit revenue of product 2 is fixed at its current value $c_2 = \$20$. What is the associated optimality range for the unit revenue for product 1, c_1, that will keep the optimum unchanged?

Answer: Substituting $c_2 = 20$ in the condition $\frac{1}{3} \leq \frac{c_1}{c_2} \leq 2$, we get

$$\frac{1}{3} \times 20 \leq c_1 \leq 2 \times 20 \text{ or } 6.67 \leq c_1 \leq 40$$

We can similarly determine the optimality range for c_2 by fixing the value of c_1 at \$30.00. Thus,

$$\left(c_2 \leq 30 \times 3 \text{ and } c_2 \geq \frac{30}{2}\right) \text{ or } 15 \leq c_2 \leq 90.$$

Algebraic Sensitivity Analysis: Using Simplex Method

Case 1: Changes in Resource Availability

Consider the following example:

TOYCO uses three operations to assemble three types of toys: **trains**, **trucks**, and **cars**.

- The daily available times for the three operations are 430, 460 , and 420 mins , respectively.
- The revenues per unit of toy train, truck, and car are \$3, \$2, and \$5, respectively.
- The assembling times per train at the three operations are 1, 3, and 1 mins, respectively.
- The assembling times per trucks and per car are (2,0,4) and (1,2,0) mins (a zero time indicates that the operation is not used).

Letting x_1, x_2, and x_3 represent the daily number of units assembled of trains, trucks, and cars, respectively, the associated LP model is given as:

$$\text{Maximinze } z = 3x_1 + 2x_2 + 5x_3$$

subject to

$$\begin{aligned}
x_1 + 2x_2 + x_3 &\leq 430 \text{ (Operation 1)} \\
3x_1 + 2x_3 &\leq 460 \text{ (Operation 2)} \\
x_1 + 4x_2 &\leq 420 \text{ (Operation 3)} \\
x_1, x_2, x_3 &\geq 0
\end{aligned}$$

Using x_4, x_5, and x_6 as the slack variables for the constraints of operations 1, 2, and 3, respectively, the optimum simplex table is obtained as

	c_j	3	2	5	0	0	0
B c_B	x_B	x_1	x_2	x_3	x_4	x_5	x_6
x_2 2	100	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0
x_3 5	230	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0
x_6 0	20	2	0	0	-2	1	1
z = 1350	Δ_j	-4	0	0	-1	-2	0

The solution recommends manufacturing 100 trucks and 230 cars but no trains. The associated revenue is $1350.

Determination of dual prices and feasibility ranges

I. Modified Primal Problem

Consider

Maximinze $z = 3x_1 + 2x_2 + 5x_3$

subject to

$$
\begin{aligned}
x_1 + 2x_2 + x_3 &\le 430 + D_1 \quad \text{(Operation 1)} \\
3x_1 + 2x_3 &\le 460 + D_2 \quad \text{(Operation 2)} \\
x_1 + 4x_2 &\le 420 + D_3 \quad \text{(Operation 3)}
\end{aligned}
$$

$$x_1, x_2, x_3 \ge 0$$

II. Convert to Standard Form

Add slack variables:

Max. $z = 3x_1 + 2x_2 + 5x_3 + 0\,x_4 + 0x_5 + 0x_6$

subject to

$$
\begin{aligned}
x_1 + 2x_2 + x_3 + x_4 &= 430 + D_1 \\
3x_1 + 2x_3 + x_5 &= 460 + D_2 \\
x_1 + 4x_2 + x_6 &= 420 + D_3
\end{aligned}
$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \ge 0$$

IV. Starting Simplex Tableau (With Capacity Changes)

	c_j	3	2	5	0	0	0	0	0	0	0
B c_B	x_B	x_1	x_2	x_3	x_4	x_5	x_6	**RHS**	D_1	D_2	D_3
x_4 0	$430+D_1$	1	2	1	1	0	0	430	1	0	0
x_5 0	$460+D_2$	3	0	2	0	1	0	460	0	1	0
x_6 0	$420+D_3$	1	4	0	0	0	1	420	0	0	1
$z = 0$	Δ_j	3	2	5	0	0	0		0	0	0

Important Observation

👉 Only RHS changes
👉 Coefficient matrix remains same

Therefore:

✓ Same simplex row operations
✓ Only RHS columns change

Step IV. Optimal Tableau After Same Iterations

	c_j	3	2	5	0	0	0	0	0	0	0
B $\mathbf{c}_B$	x_B	x_1	x_2	x_3	x_4	x_5	x_6	**RHS**	D_1	D_2	D_3
x_2 2	$100+\frac{1}{2}D_1-\frac{1}{4}D_2$	$-1/4$	1	0	1/2	-1/4	0	100	1/2	-1/4	0
x_3 5	$230+\frac{1}{2}D_2$	$3/2$	0	1	0	1/2	0	230	0	1/2	0
x_6 0	$20-2D_1+D_2+D_3$	2	0	0	-2	1	1	20	-2	1	1
z =1350	Δ_j	-4	0	0	-1	-2	0	0	-1	-2	0

- The coefficients of D_1, D_2, and D_3 in the optimal z-row are exactly those of the slack variables x_4, x_5, and x_6.
- This means that the dual prices equal the negative of the coefficients of the slack variables in the optimal z –row (Δ_j –row).

The new optimum tableau provides the following optimal solution:

$$z = 1350 + D_1 + 2D_2$$

$$x_2 = 100 + \frac{1}{2}D_1 - \frac{1}{4}D_2$$

$$x_3 = 230 + \frac{1}{2}D_2$$

$$x_6 = 20 - 2D_1 + D_2 + D_3$$

Dual Prices

- The value of the objective function can be written as

$$z = 1350 + 1D_1 + 2D_2 + 0D_3$$

- This shows that the corresponding dual prices for operations 1, 2, and 3 are 1, 2, and 0 ($/min), respectively. This means

a) A unit change in operation 1 capacity ($D_1 = \pm 1$ min) changes z by \$1.

b) A unit change in operation 2 capacity ($D_2 = \pm 1$ min) changes z by \$2.

c) A unit change in operation 3 capacity ($D_3 = \pm 1$ min) changes z by \$0.

Feasibility range

The current solution remains feasible if all the basic variables remain nonnegative, i.e.,

$$x_2 = 100 + \frac{1}{2}D_1 - \frac{1}{4}D_2 \geq 0$$

$$x_3 = 230 + \frac{1}{2}D_2 \geq 0$$

$$x_6 = 20 - 2D_1 + D_2 + D_3 \geq 0$$

- Simultaneous changes D_1, D_2, and D_3 that satisfy these inequalities will keep the solution feasible.

- The new optimum solution can be found by substituting out the values of D_1, D_2, and D_3.

Suppose that the manufacturing time available for operations 1, 2, and 3 are 480, 440, and 400 mins, respectively.

- Then, $D_1 = 480 - 430 = 50, D_2 = 440 - 460 = -20$, and $D_3 = 400 - 420 = -20$.
- Substituting in the feasibility conditions, we get

$$x_2 = 130 > 0 \quad \text{(feasible)}$$
$$x_3 = 220 > 0 \quad \text{(feasible)}$$
$$x_6 = -110 < 0 \quad \text{(infeasible)}$$

- As $x_6 < 0$, hence the current solution does not remain feasible.

Suppose that the manufacturing time available for operations 1, 2, and 3 are 400, 448, and 430 mins, respectively.

- Then, $D_1 = -30, D_2 = -12$, and $D_3 = 10$.
- Substituting in the feasibility conditions, we get

$$x_2 = 88 > 0 \quad \text{(feasible)}$$
$$x_3 = 224 > 0 \quad \text{(feasible)}$$
$$x_6 = 78 > 0 \quad \text{(feasible)}$$

- The new (optimal) feasible solution is

$$x_1 = 88, x_3 = 224, \text{ and } x_6 = 68$$

with $\boldsymbol{z} = 3(0) + 2(88) + 5(224) = \$\mathbf{1296} = 1350 + 1(-30) + 2(-12) + 0(10)$.

- The given conditions can produce the individual feasibility ranges associated with changing the resources one at a time.
- For example, a change in operation 1 time only means that $D_2 = D_3 = 0$. The simultaneous conditions thus reduce to:

$$\left.\begin{array}{r} x_2 = 100 + \frac{1}{2}D_1 \geq 0 \Rightarrow D_1 \geq -200 \\ x_3 = 230 > 0 \\ x_6 = 20 - 2D_1 \geq 0 \Rightarrow D_1 \leq 10 \end{array}\right\} \Rightarrow -200 \leq D_1 \leq 10$$

This means that the dual price for operation 1 is valid in the feasibility range: $-200 \leq D_1 \leq 10$.

- Similarly, feasibility ranges for operations 2 and 3 are: $-20 \leq D_2 \leq 400$ and $-20 \leq D_3 \leq \infty$.

Dual prices and their feasibility ranges for the TOYCO model:

Resource	Dual price($)	Feasibility range	Resource amount (minutes)		
			Minimum	Current	Maximum
Operation 1	1	$-200 \le D_1 \le 10$	230	430	440
Operation 2	2	$-20 \le D_2 \le 400$	420	440	840
Operation 3	0	$-20 \le D_3 < \infty$	400	420	∞

NOTE: The dual prices will remain applicable for any simultaneous changes that keep the solution feasible, even if the changes violate the individual ranges.

- For example, consider the changes $D_1 = 30, D_2 = -12$, and $D_3 = 100$. Then

$$x_2 = 100 + \frac{1}{2}(30) - \frac{1}{4}(-12) = 118 > 0 \qquad \text{(feasible)}$$

$$x_3 = 230 + \frac{1}{2}(-12) = 224 > 0 \qquad \text{(feasible)}$$

$$x_6 = 20 - 2(30) + (-12) + (100) = 48 > 0 \qquad \text{(feasible)}$$

- Even though $D_1 = 30$ violates the feasibility range $-200 \le D_1 \le 10$, the changes will keep the solution feasible.
- This means that the dual prices will remain applicable, and we can compute the new optimum objective value from the dual prices as

$$z = 1350 + 1(30) + 2(-12) + 0(100) = \$1356.$$