OPERATIONS RESEARCH

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The Assignment Problem

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- The assignment problem is a special type of linear programming problem in which a number of jobs are assigned to an equal number of persons so as to minimize the total cost.

- It **can** be solved by the Simplex method.

- However, the **Assignment Method** is computationally more efficient and tailored for assignment problems.

- Common Examples: Assigning workers to jobs, drivers to trucks, machines to tasks, salespeople to territories, and research teams to projects.

The Nature of the Assignment Problem

Problem setup: Let there be n jobs to be performed by n persons. We assume:

- Each person can potentially perform each job with different costs or efficiencies.
- Each person is assigned exactly one job.
- Each job is assigned to exactly one person.

Cost matrix: Let c_{ij} be the cost of assigning person i to job j. The data is arranged in an $n \times n$ cost (or effectiveness) matrix $C = [c_{ij}]$.

Objective: Find an assignment so that the total cost for performing all jobs is minimum.

Mathematical Formulation

Objective: Minimize the total cost

$$Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} ,$$

$$\text{where } x_{ij} = \begin{cases} 1, & \text{If the Person } i \text{ is assigned to Job } j \\ 0, & \text{Otherwise.} \end{cases}$$

Subject to the conditions

- $\sum_{j=1}^{n} x_{ij} = 1$ (Only one job is done by the i-th person for $i = 1, \dots, n$).
- $\sum_{i=1}^{n} x_{ij} = 1$ (Only one person should be assigned to the j-th job for $j = 1, \dots, n$).

Fundamental Theorems

Reduction Theorem: Adding or subtracting a constant from every element of a row or column of a cost matrix $[\mathrm{c_{ij}}]$ does not change which assignment is optimal.

Row/column reductions preserve optimality.

Zero-optimality Theorem: If all costs $\mathrm{c_{ij}} \geq 0$ and a feasible assignment $x_{\mathrm{ij}} = \mathrm{X_{ij}}$ attains total cost Z = 0, that assignment is optimal, i.e., the solution minimizes the objective function.

The minimum possible value that Z can attain is 0.

Assignment Algorithm (Hungarian Assignment Method)

The Hungarian method uses the systematic row and column reductions, then finds a complete zero-based assignment using line covering and matrix modification until an optimal assignment emerges.

Step 1: Row and Column Reduction.

a) Row reduction: Subtract the minimum element of each row of the cost matrix, from the elements of the respective rows.

b) Column reduction: In the resulting matrix, subtract he minimum element of each column from all the elements of the respective columns.

Hungarian Assignment Method

Step 2: Make ZERO assignments.

If complete assignment is possible → the assignment is optimal.

If not, we need to modify the cost matrix to create some more zeros in it.

Procedure to mark assignments using zeros:

i) Find the first row with exactly one zero element→ mark assignment (☐) there and cross (✗) out other zeros in that column.

ii) Continue the same procedure until all the rows have been examined.

iii) Repeat the same for columns in the resulting matrix (columns with one unmarked zero).

iii) Continue alternating until no new **single-zero** rows/columns exist.

If all the zeros have been marked (☐ or ✗) we have a maximal assignment.

If we obtain n assignments, i.e., an assignment (one $\boxed{0}$) in every row and every column, we have achieved an optimal solution.

Unmarked zero left in the resulting matrix: In this case, we have more than one zeros in each row and column. Need to proceed with hit and trial method to break up such tie of zeros.

Hungarian Assignment Method

Step 3: Covering Zeros with Minimum number of Lines.

In step 2, if every row and every column of the matrix do not contain assignment, then we cover all zeros with the minimum number of horizontal and vertical lines.

Procedure to draw minimum number of lines:

i) Mark (✓) all rows that have no assigned zero ($\boxed{0}$).

ii) Mark (✓) columns that contains a zero in a marked row.

iii) Mark (✓) rows (not already marked) that has an assignment in a marked column.

iv) Repeat steps (ii)–(iii) until no new marks (✓) appear.

v) Draw lines through all unmarked rows and marked columns → this is the minimal cover.

NOTE 1. The minimum number of lines required to pass through all the zeros of the matrix is the same as the maximum number of assigned independent zeros of the matrix.

NOTE 2. These lines cover all the zeros, and each line passes through one and only one assigned zero.

NOTE 3. If the number of these lines is n, an optimal assignment exists among the zeros; otherwise proceed to Step 4 for the complete assignment.

Hungarian Assignment Method

Step 4: Modifying the Matrix to Create More Zeros

i) Select the smallest of the elements not covered by any line. Let it be h.

ii) Subtract h from all uncovered elements.
iii) Add h to to every element that lies at the intersection of two lines.
iv) Leave covered-but-not-intersected elements unchanged.

These operations ((ii)-(iv)) increase the number of zeros without changing the optimal assignment.

v) Return to Step 2 and proceed further.

vi) Continue the process until n assignments are achieved.

Example1

A department head has four subordinates, and four tasks have to be performed. Subordinates differ in efficiency and tasks differ in their intrinsic difficulty. The time each man would take to perform each task is given in the given effectiveness matrix.

How should the tasks be allocated, one to a man, so as to minimize the total man hours?

Subordinates/ Task	**I**	**II**	**III**	**IV**
A	8	26	17	11
B	13	28	4	26
C	38	19	18	15
D	19	26	24	10

Step 1: Subtracting the minimum element of each row from every element of the corresponding row, the matrix reduces to corresponding column, and then subtracting the minimum element of each column of the resulting matrix we get

	I	II	III	IV
A	8	26	17	11
B	13	28	4	26
C	38	19	18	15
D	19	26	24	10

	I	II	III	IV
A	0	18	9	3
B	9	24	0	22
C	23	4	3	0
D	9	16	14	0

	I	II	III	IV
A	0	14	9	3
B	9	20	0	22
C	23	0	3	0
D	9	12	14	0

Step 2: Starting with row 1 of the reduced matrix, we examine the row one by one until a row containing only one zero element is found. We mark □ at this zero and a cross ✗ over all zeros lying in the column containing the assigned zero. Continue in this manner until all the rows have been examined. After that do the same procedure for columns.

	I	II	III	IV
A	$\boxed{0}$	14	9	3
B	9	20	$\boxed{0}$	22
C	23	0	3	$\cancel{0}$
D	9	12	14	$\boxed{0}$

	I	II	III	IV
A	$\boxed{0}$	14	9	3
B	9	20	$\boxed{0}$	22
C	23	$\boxed{0}$	3	$\cancel{0}$
D	9	12	14	$\boxed{0}$

- In the resulting matrix all zeros have been either assigned or crossed.
- Every row and every column have an assignment.
- Hence an optimal solution has been obtained.
- The optimal solution is
  A →I, B → III, C→II, D→IV.
- The minimum total man hours is
  8+4+19+10=41

(The original cost matrix is used for calculating the minimum cost.)

Example 2

Solve the minimal assignment problem whose effectiveness matrix is

	I	II	III	IV
A	2	3	4	5
B	4	5	6	7
C	7	8	9	8
D	3	5	8	4

Step 1: Row reduction

	I	II	III	IV
A	0	1	2	3
B	0	1	2	3
C	0	1	2	1
D	0	2	5	1

Column reduction

	I	II	III	IV
A	0	0	0	2
B	0	0	0	2
C	0	0	0	0
D	0	1	3	0

Step 2: Assignments using zeros:

- Here none of the rows or columns contain exactly one zero, therefore we start with row 1 searching two zeros.
- Row 4 has two zeros.
- Arbitrarily make an assignment to one of these two zeros, say zero in the column 1
- Cross other zeros in row 4 and column 1.

	I	**II**	**III**	**IV**
A	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	2
B	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	2
C	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$
D	$\boxed{0}$	1	3	$\cancel{0}$

- Now we examine the columns.
- Column 4 contains only one unmarked zero in row 3.
- We make assignment at this zero and cross all other zeros of this row.
- We again check the rows and columns for **one** unmarked zero.
- There is no such row or column.
- Start with row 1 searching two unmarked zeros.
- Row 1 contains two zeros.
- We can make an assignment at any one of these zeros.

- Make assignment at zero of column 2, and cross other zeros of row 1 and column 2.
- Now the second row contains only one unmarked zero in third column.
- We make an assignment there.

We get the following optimal assignment :

A →II, B →III, C→IV, D →I.

Minimum cost = 3 + 6 + 8 + 3 = 20.

NOTE: In this example other optimal assignments are also possible. Each will have the cost 20.

Example 3

A car hire company has one car at each of five depots a, b, c, d, and e. A customer requires a car in each town namely A, B , C, D, and E. Distance (in kms) between depots (origins) and towns (destinations) are given in the given distance matrix.

How should cars be assigned to customers so as to minimize the distance travelled?

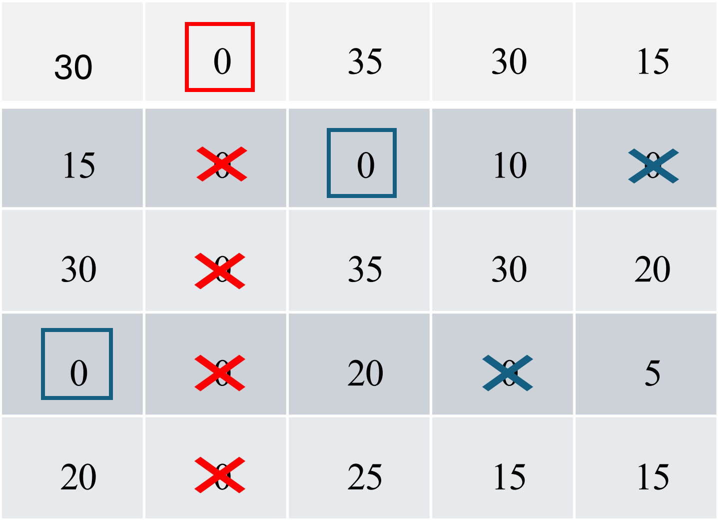
	a	b	c	d	e
A	160	130	175	190	200
B	135	120	130	160	175
C	140	110	155	170	185
D	50	50	80	80	110
E	55	35	70	80	105

Step 1: Row and Column Reduction

30	0	45	60	70
15	0	10	40	55
30	0	45	60	75
0	0	30	30	60
20	0	35	45	70

30	0	35	30	15
15	0	0	10	0
30	0	35	30	20
0	0	20	0	5
20	0	25	15	15

Step 2: Make zero assignments

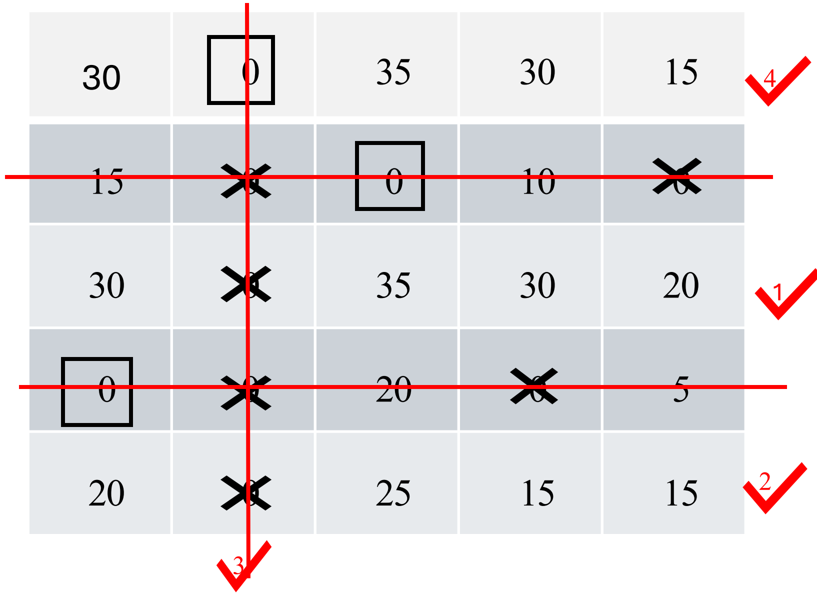


30	$\boxed{0}$	35	30	15
15	$\cancel{0}$	$\boxed{0}$	10	$\cancel{0}$
30	$\cancel{0}$	35	30	20
$\boxed{0}$	$\cancel{0}$	20	$\cancel{0}$	5
20	$\cancel{0}$	25	15	15

Number of assigned zero = 3 < 5

Step 3: Covering Zeros with Lines

- Mark row 3 and row 5 as having no assignment.
- Mark column 2 as having zeros in marked row 3 (and 5).
- Mark row 1 as it contains assignment in the marked columns 2.
- Draw lines through marked column 2 and unmarked rows 2 and 4.



30	$\boxed{0}$	35	30	15
15	$\cancel{0}$	$\boxed{0}$	10	$\cancel{0}$
30	$\cancel{0}$	35	30	20
$\boxed{0}$	$\cancel{0}$	20	$\cancel{0}$	5
20	$\cancel{0}$	25	15	15

Step 4: Modify the Matrix

- Smallest element among all uncovered elements of the matrix in step 3 is 15.
- Subtract 15 from all the elements that do not have a line through them.
- Add 15 to every element that lies at the intersection of two lines.
- Remaining elements unchanged.

15	0	20	15	0
15	15	0	10	0
15	0	20	15	5
0	15	20	0	5
5	0	10	0	0

Now proceed with this matrix.

Step 5: Again perform step 2 and make the zero assignments.

15	$\cancel{0}$	20	15	$\boxed{0}$
15	15	$\boxed{0}$	10	$\cancel{0}$
15	$\boxed{0}$	20	15	5
$\boxed{0}$	15	20	$\cancel{0}$	5
5	$\cancel{0}$	10	$\boxed{0}$	$\cancel{0}$

Observation:

- There are no remaining zeros.
- Every row (column) has an assignment.

Conclusion:

The complete optimal assignment plan is obtained.

Optimal assignment plan is:
A→e, B→c, C→b,
D→a, E→ d.

The minimum cost (distance travelled) =

200 + 130 + 110 + 50 + 80

= 570 kms.

Example 4

In an airline that operates seven days a week, has the given time-table. Crews must have a minimum layover of 5 hours between flights. Obtain the pair of flights that minimizes layover time away from home.

For a given pair, the crew will be based at the city that results in the smallest layover. For each pair, mention the the town where the crew should be based.

Delhi	to	Jaipur
Flight No.	Departure	Arrival
1	7:00 AM	8:00 AM
2	8:00 AM	9: 00 AM
3	1:30 AM	2:30 PM
4	6:30 PM	7:30 PM

Jaipur	to	Delhi
Flight No.	Departure	Arrival
101	8:00 AM	9:15 AM
102	8:30 AM	9: 45 AM
103	12:00 Noon	1:15 PM
104	5:30 PM	6:45 PM

First construct the tables for layover times between the flights.

- Suppose flight no. 1 is paired with flight no. 103 when crew is based at Delhi.
- The time of stay at Jaipur will be the layover time away from home.
- Flight no. 1 which reaches Jaipur at 8:00 AM, cannot fly at 12:00 Noon on the same day as minimum layover time is 5 hours.
- It will depart Jaipur on the next day which will result in a layover time of 28 hours.
- Similarly, other layover times can be calculated.

When crew based at Delhi

	101	**102**	**103**	**104**
1	24	24.5	28	9.5
2	23	23.5	27	8.5
3	17.5	18	21.5	27
4	12.5	13	16.5	22

When crew based at Jaipur

	101	**102**	**103**	**104**
1	21.75	21.25	17.75	12.25
2	22.75	22.25	18.75	13.25
3	28.25	27.75	24.25	18.75
4	9.25	8.75	5.25	23.75

- To avoid the fractions, we measure the layover times in terms of quarter hour (0.25 hr. or 15 minutes) as one unit of time.

- Multiplying the tables, the modified tables by 4 we get:

When crew based at Delhi

	101	**102**	**103**	**104**
1	96	98	112	38
2	92	94	108	34
3	70	72	86	108
4	50	52	66	88

When crew based at Jaipur

	101	**102**	**103**	**104**
1	87	85	71	49
2	91	89	75	53
3	113	111	97	75
4	37	35	21	95

- Next, combine the tables, choosing that base which gives a **lesser** layover time for each pairing.

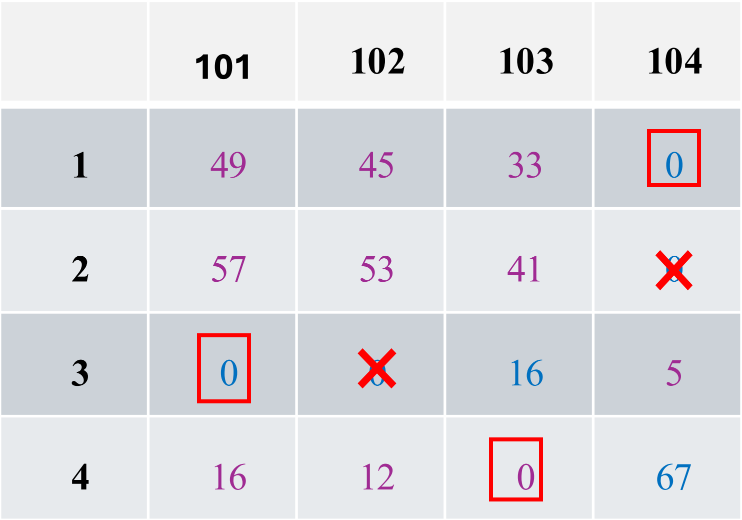
Minimum layover time-table

	101	102	103	104
1	87	85	71	38
2	91	89	75	34
3	70	72	86	75
4	37	35	21	88

- Now solve it by usual assignment-technique.
- Step I: After row and column reduction, we get

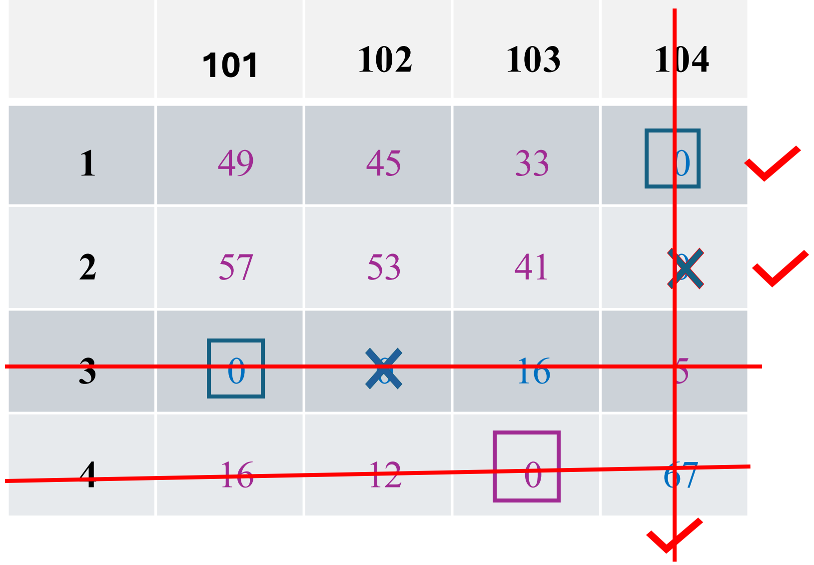
	101	102	103	104
1	49	45	33	0
2	57	53	41	0
3	0	0	16	5
4	16	12	0	67

- Step II: Making the 0 assignment:



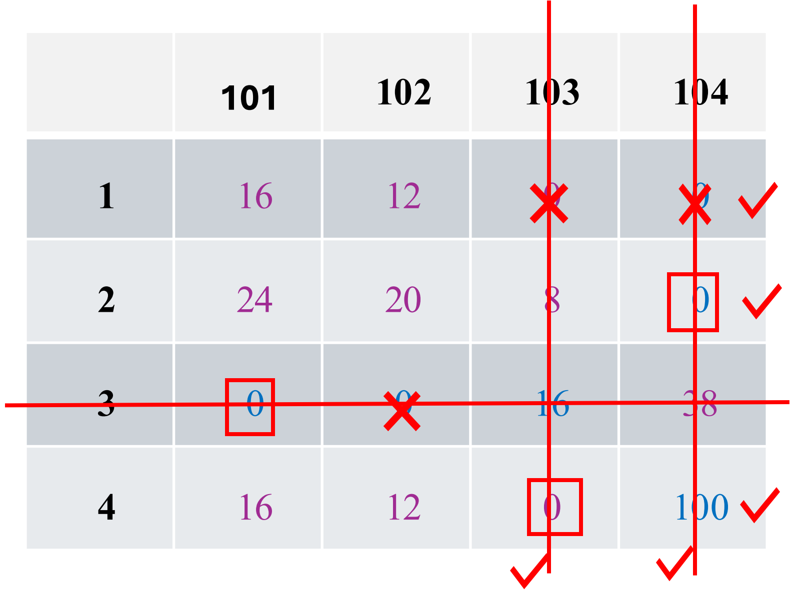
	101	102	103	104
1	49	45	33	0
2	57	53	41	0
3	0	0	16	5
4	16	12	0	67

- Step III: Covering Zeros with Lines:



	101	102	103	104
1	49	45	33	0
2	57	53	41	0
3	0	0	16	5
4	16	12	0	67

- Step IV: Obtaining modified matrix and proceeding with similar assignment process:



	101	102	103	104
1	16	12	0	0
2	24	20	8	0
3	0	0	16	38
4	16	12	0	100

We get the second time modified matrix:

	101	102	103	104
1	4	0	0	0
2	12	8	8	0
3	0	0	28	50
4	4	0	0	100

Step V: Making the zero assignment in obtained matrix:

	101	102	103	104
1	4	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$
2	12	8	8	$\boxed{0}$
3	$\boxed{0}$	$\cancel{0}$	28	50
4	4	$\cancel{0}$	$\boxed{0}$	100

OR

	101	102	103	104
1	4	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$
2	12	8	8	$\boxed{0}$
3	$\boxed{0}$	$\cancel{0}$	28	50
4	4	$\boxed{0}$	$\cancel{0}$	100

From the above tables, two optimal assignments are:

(i) (1 → 102), (2 → 104), (3 → 101), (4 → 103).

(ii) (1 → 103), (2 → 104), (3 → 101), (4 → 102).

Purple denote that crew is based at Jaipur.

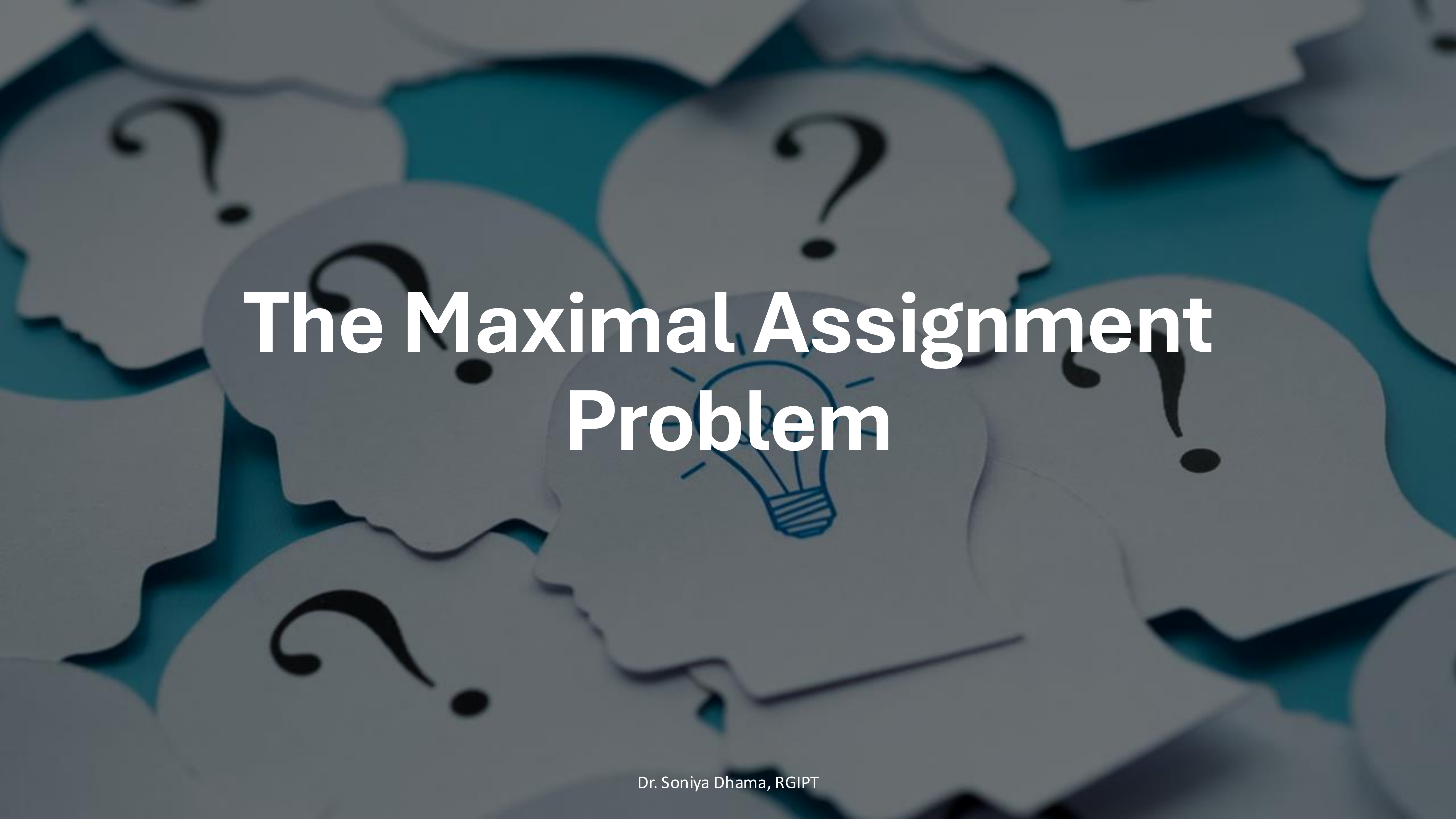
Blue denote that crew is based at Delhi.

In both the cases minimum layover time is 210 quarter hours i.e., **52 hours 30 minutes**.

Practice Problem

Solve the assignment problem represented by the given matrix.

	I	II	III	IV	V	VI
A	9	22	58	11	19	27
B	43	78	72	50	63	48
C	41	28	91	37	45	33
D	74	42	27	49	39	32
E	36	11	57	22	25	18
F	3	56	53	31	17	28



The Maximal Assignment Problem

Assignment Problem with Maximization

- Sometimes the assignment problem aims to maximize an objective function instead of minimizing it.
- For example, the objective may be to assign persons to jobs to obtain maximum profit.
- Such problems can be solved by converting the maximization problem into a minimization problem.
- After conversion, the usual assignment algorithm (Hungarian method) can be applied.
- The conversion is done by transforming the profit matrix into an equivalent cost matrix.

Conversion of Profit Matrix to Cost Matrix

Method 1: Using the Largest Element

- Select the largest element in the given profit matrix.
- Subtract each element of the matrix from this largest element.

Method 2: Using Negative Values

- Place a minus sign before each element of the profit matrix.

Example

A company has 5 jobs to be done. The given matrix shows the return in rupees on assigning ith $(i = \text{I}, \text{II}, \text{III}, \text{IV}, \text{V})$ machine to the jth job (j= A, B, C, D, E).

Assign the five jobs to the five machines so as to **maximize** the total expected profit.

	I	**II**	**III**	**IV**	**V**
A	5	11	10	12	4
B	2	4	6	3	5
C	3	12	5	14	6
D	6	14	4	11	7
E	7	9	8	12	5

- First convert the problem from maximization to minimization
- The greatest element of the given matrix is 14.
- Subtracting all the elements of the given matrix from 14, the modified matrix can be obtained as

9	3	4	2	10
12	10	8	11	9
11	2	9	0	8
8	0	10	3	7
7	5	6	2	9

- Now follow the usual procedure of solving an assignment problem.

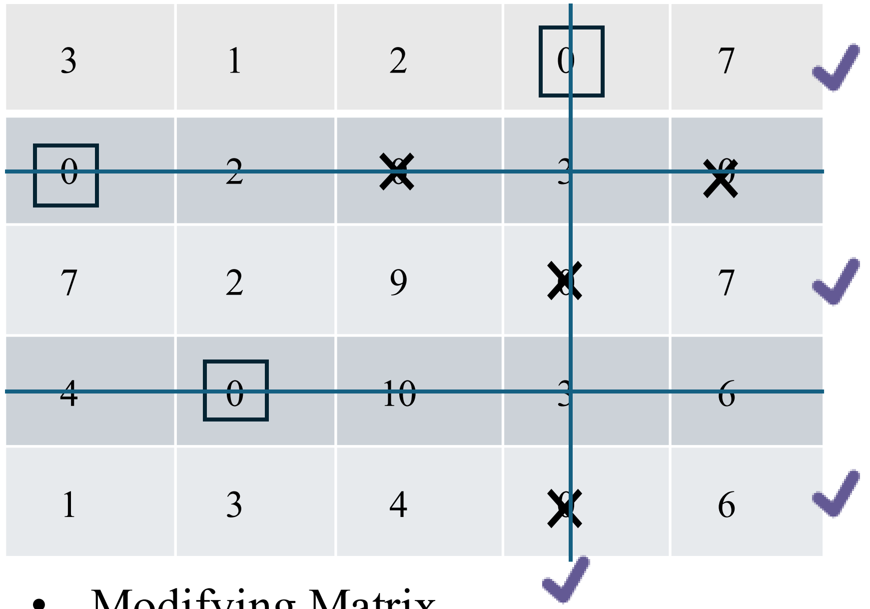
- Row and Column Reduction

3	1	2	0	7
0	2	0	3	0
7	2	9	0	7
4	0	10	3	6
1	3	4	0	6

- Giving zero assignments

3	1	2	$\boxed{0}$	7
$\boxed{0}$	2	$\cancel{0}$	3	$\cancel{0}$
7	2	9	$\cancel{0}$	7
4	$\boxed{0}$	10	3	6
1	3	4	$\cancel{0}$	6

- Covering zeros



3	1	2	$\boxed{0}$	7	✓
$\boxed{0}$	2	$\cancel{0}$	3	$\cancel{0}$	
7	2	9	$\cancel{0}$	7	✓
4	$\boxed{0}$	10	3	6	
1	3	4	$\cancel{0}$	6	✓
			✓		

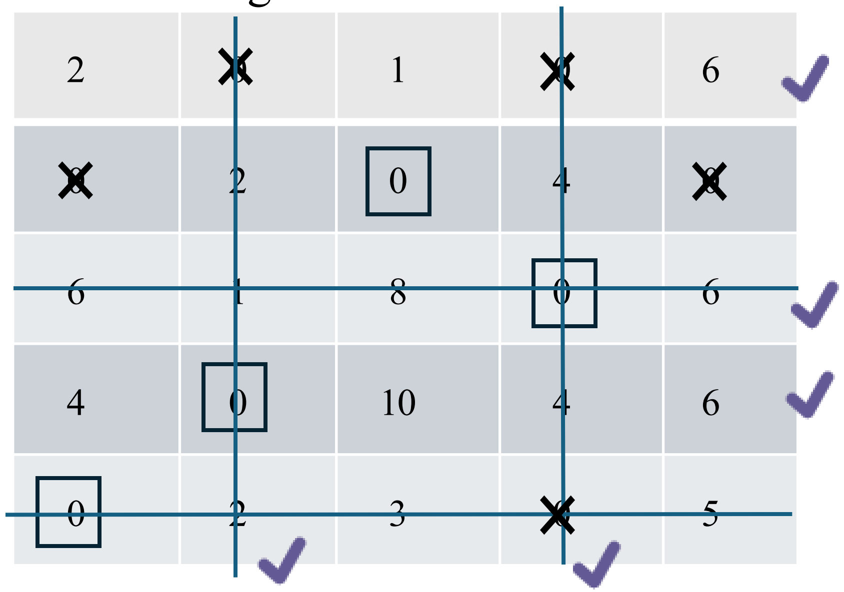
- Modifying Matrix

2	0	1	0	6
0	2	0	4	0
6	1	8	0	6
4	0	10	3	6
0	2	3	0	5

- Giving zero assignments

2	$\cancel{0}$	1	$\cancel{0}$	6
$\cancel{0}$	2	$\boxed{0}$	4	$\cancel{0}$
6	1	8	$\boxed{0}$	6
4	$\boxed{0}$	10	4	6
$\boxed{0}$	2	3	$\cancel{0}$	5

- Covering zeros



2	$\cancel{0}$	1	$\cancel{0}$	6	✓
$\cancel{0}$	2	$\boxed{0}$	4	$\cancel{0}$	
6	1	8	$\boxed{0}$	6	✓
4	$\boxed{0}$	10	4	6	✓
$\boxed{0}$	2	3	$\cancel{0}$	5	
	✓		✓		

- Modifying Matrix

1	0	0	0	5
0	3	0	5	0
5	1	7	0	5
3	0	9	4	5
0	3	3	1	5

- Giving zero assignments

1	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	5
$\cancel{0}$	3	$\cancel{0}$	5	$\boxed{0}$
5	1	7	$\boxed{0}$	5
3	$\boxed{0}$	9	4	5
$\boxed{0}$	3	3	1	5

- The optimal assignment of jobs to maximize the profit is: Machine → job: 1→C, 2→E, 3→D, 4→B, 5 → A.
- From the given matrix, the maximum profit = 10 + 5 + 14 + 14 + 7 = Rs. 50

Example

A company has four territories open and four salesmen available for assignments. The territories are not equally rich in their sales potential; it is estimated that a typical salesman operating in each territory would bring in the following annual sales:

Territory	I	II	III	IV
Annual Sales	60,000	50,000	40,000	30,000

Four salesmen are also considered to differ in their ability: it is estimated that, working under the same conditions, their yearly sales would be proportionately as follows :

Salesman	A	B	C	D
Proportion	7	5	5	4

If the criterion is maximum expected total sales, then intuitive answer is to assign the best salesman to the richest territory, the next best salesman to the second richest, and so on. Verify this answer by the assignment technique.

- We first construct the effectiveness matrix.
- The sum of proportions of sales of four salesmen $= 7+5+5+4 = \mathbf{21}$.
- Consider Rs 10,000 as one unit.
- The annual sales in the four territories by the four salesmen are given as:

$$\mathbf{A:}\ \frac{7}{21}\times 6,\ \frac{7}{21}\times 5,\ \frac{7}{21}\times 4\ \frac{7}{21}\times 3 = \frac{\mathbf{42}}{\mathbf{21}},\ \frac{\mathbf{35}}{\mathbf{21}},\ \frac{\mathbf{28}}{\mathbf{21}},\ \frac{\mathbf{21}}{\mathbf{21}}$$

$$\mathbf{B:}\ \frac{5}{21}\times 6,\ \frac{5}{21}\times 5,\ \frac{5}{21}\times 4,\ \frac{5}{21}\times 3 = \frac{\mathbf{30}}{\mathbf{21}},\ \frac{\mathbf{25}}{\mathbf{21}},\ \frac{\mathbf{20}}{\mathbf{21}},\ \frac{\mathbf{15}}{\mathbf{21}}$$

$$\mathbf{C:}\ \frac{5}{21}\times 6,\ \frac{5}{21}\times 5,\ \frac{5}{21}\times 4,\ \frac{5}{21}\times 3 = \frac{\mathbf{30}}{\mathbf{21}},\ \frac{\mathbf{25}}{\mathbf{21}},\ \frac{\mathbf{20}}{\mathbf{21}},\ \frac{\mathbf{15}}{\mathbf{21}}$$

$$\mathbf{D:}\ \frac{4}{21}\times 6,\ \frac{4}{21}\times 5,\ \frac{4}{21}\times 4,\ \frac{4}{21}\times 3 = \frac{\mathbf{24}}{\mathbf{21}},\ \frac{\mathbf{20}}{\mathbf{21}},\ \frac{\mathbf{16}}{\mathbf{21}},\ \frac{\mathbf{12}}{\mathbf{21}}$$

- To avoid fractional values, we consider the sales in 21 years.

- The maximum sale matrix is obtained as follows

	I	II	III	IV
A	42	35	28	21
B	30	25	20	15
C	30	25	20	15
D	24	20	16	12

- This is a 'maximization' problem.
- To convert it into a 'minimization' multiply each element of the above matrix by -1.

-42	-35	-28	-21
-30	-25	-20	-15
-30	-25	-20	-15
-24	-20	-16	-12

- Now, we solve the minimization problem by usual assignment algorithm.

- Row Reduction

0	7	14	21
0	5	10	15
0	5	10	15
0	4	8	12

- Column Reduction

0	3	6	9
0	1	2	3
0	1	2	3
0	0	0	0

- Giving zero assignments

$\boxed{0}$	3	6	9
$\cancel{0}$	1	2	3
$\cancel{0}$	1	2	3
$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$

- Covering zeros with lines

$\boxed{0}$	3	6	9	✓
$\cancel{0}$	1	2	3	✓
$\cancel{0}$	1	2	3	✓
$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	
✓				

- Modifying the matrix

0	2	5	8
0	0	1	2
0	0	1	2
1	0	0	0

- Giving zero assignments and covering zeros

$\boxed{0}$	2	5	8	✓
$\cancel{0}$	$\boxed{0}$	1	2	✓
$\cancel{0}$	$\cancel{0}$	1	2	✓
1	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	
✓	✓			

- Modifying the matrix

0	2	4	7
0	0	0	1
0	0	0	1
2	1	0	0

- Giving zero assignments

$\boxed{0}$	2	4	7
$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	1
$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	1
2	1	$\cancel{0}$	$\boxed{0}$

$\boxed{0}$	2	4	7
$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	1
$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	1
2	1	$\cancel{0}$	$\boxed{0}$

- Two optimal solutions are
  (i) A→I, B→II, C→III, D→IV
  (ii) A →I, B→III, C→II, D →IV

Conclusion:

- The best salesman A is assigned to the richest territory I
- The worst salesman D to the poorest territory IV.
- Salesmen B and C being equally good, so they may be assigned to either II or III.

Unbalanced Assignment Problem.

Unbalanced assignment problem: The assignment problem in which the number of tasks (jobs) is not equal to the number of facilities (persons).

- The cost matrix of an unbalanced assignment problem is not a square matrix.

Method To Solve:

- **Add dummy rows or columns** to the cost matrix to form it a square matrix.
- The costs of dummy entries are taken as zero.
- Apply the usual assignment algorithm (Hungarian Method).

Example

	1	**2**	**3**
I	9	26	15
II	13	27	6
III	35	20	15
IV	18	30	20

A department head has four tasks to be performed and three subordinates. The subordinates differ in efficiency. The estimates of the time, each subordinate would take to perform, are given in the matrix.

How should he allocate the tasks, one to each man, so as to minimize the total man hours?

- Matrix is not square ⇒ Unbalanced assignment problem.
- Introduce one dummy subordinate (4th column with zero costs).

	1	**2**	**3**	**4**
I	9	26	15	0
II	13	27	6	0
III	35	20	15	0
IV	18	30	20	0

- Solve by usual (Hungarian) method.

- Row and Column reduction

	1	**2**	**3**	**4**
I	0	6	9	0
II	4	7	0	0
III	26	0	9	0
IV	9	10	11	0

- Giving zero assignments

	1	**2**	**3**	**4**
I	$\boxed{0}$	6	9	$\cancel{0}$
II	4	7	$\boxed{0}$	$\cancel{0}$
III	26	$\boxed{0}$	9	$\cancel{0}$
IV	9	10	11	$\boxed{0}$

- Each row and column have zero assignments.
- Hence the optimal assignment is as follows.

Tasks ⟶ Subordinates:

I ⟶1, II ⟶3, III ⟶2.

Task IV remains unassigned.

Restrictions On Assignment

- Sometimes due to some restrictions the assignment of a particular facility to a particular job is not permitted.

- To overcome such difficulty we assign a very high cost (say, infinite cost) to the corresponding cell.

- The activity will be automatically excluded from the optimal solution.

Example

Four engineers are available to design four projects. Engineer 2 is not competent to design the project B. Given the following time estimates needed to each engineer to design a given project, find how should the engineers be assigned to projects so as to minimize the total design time of four projects.

Projects

Engineers

	A	B	C	D
1	12	10	10	8
2	14	Not suitable	15	11
3	6	10	16	4
4	8	10	9	7

Solution

- To avoid the assignment 2 → B, take its time to be very large.

	A	B	C	D
1	12	10	10	8
2	14	∞	15	11
3	6	10	16	4
4	8	10	9	7

- Now, solve by usual method.

- Row and Column Reduction

	A	B	C	D
1	3	0	0	0
2	2	∞	2	0
3	1	4	10	0
4	0	1	0	0

- Giving zero assignments and covering zeros

	A	B	C	D	
1	3	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	
2	2	∞	2	$\boxed{0}$	✓
3	1	4	10	$\cancel{0}$	✓
4	$\boxed{0}$	1	$\cancel{0}$	$\cancel{0}$	
				✓	

- Modifying Matrix

	A	B	C	D
1	3	0	0	1
2	1	∞	1	0
3	0	3	9	0
4	0	1	0	1

- Giving zero assignments

	A	B	C	D
1	3	$\boxed{0}$	$\cancel{0}$	1
2	1	∞	1	$\boxed{0}$
3	$\boxed{0}$	3	9	$\cancel{0}$
4	$\cancel{0}$	1	$\boxed{0}$	1

- Each row and each column have a zero assignment.

- The optimal assignment (Engineer → Project) is:

1 → B, 2→ D, 3 →A, 4 → C.

- From the given matrix total minimum time = 10+11 + 6 + 9 = 36.