OPERATIONS RESEARCH

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Example1

$$\text{Maximize } Z = 40x_1 + 35x_2$$

subject to

$$2x_1 + 3x_2 \leq 60$$
$$4x_1 + 3x_2 \leq 96$$
$$x_1, x_2 \geq 0$$

Nature of the Problem

- The given problem is a **maximization problem**
- All RHS constants b_i are **positive**
- Hence, the problem is suitable for the **simplex method**

Standard Form

Introduce slack variables s_3, s_4:

$$2x_1 + 3x_2 + s_3 = 60$$
$$4x_1 + 3x_2 + s_4 = 96$$

Objective function becomes:

$$\text{Maximize } Z = 40x_1 + 35x_2 + 0s_3 + 0s_4$$
$$x_1, x_2, s_3, s_4 \geq 0$$

Initial Basic Feasible Solution

- Set non-basic variables:

$$x_1 = 0, x_2 = 0$$

- From constraints:

$$s_3 = 60, s_4 = 96$$

Initial BFS:

$$(x_1, x_2, s_3, s_4) = (0,0,60,96)$$

First Simplex Table

B	c_B	c_j	40	35	0	0	Min. ratio
		$\mathbf{x}_B$	Y_1	Y_2	$Y_3(\beta_1)$	$Y_4(\beta_2)$	x_B/Y_1
Y_3	0	60	2	3	1	0	$60/2 = 30$
Y_4	0	96	4	3	0	1	$96/4 = 24$ (min)
$Z = c_B x_B = 0$		$\Delta_j = c_j - c_B Y_j$	40 (max)	35	0	0	

- Since some $\Delta_j > 0$,solution is **not optimal.**
- **Entering variable:** x_1)largest Δ_j)

- Minimum ratio test:

$$\frac{x_B}{Y_1} = \left(\frac{60}{2}, \frac{96}{4}\right) = (30,24)$$

- Minimum = 24 ⟶ **Leaving variable:** s_4
- **Pivot element:** 4

Second Simplex Table

B	$\mathbf{c}_B$	c_j	40	35	0	0	Min. ratio
		$\mathbf{x}_B$	$Y_1(\beta_2)$	Y_2	$Y_3(\beta_1)$	Y_4	$\mathbf{x}_B/Y_2$
Y_3	0	12	0	3/2	1	-1/2	8 (min)
Y_1	40	24	1	3/4	0	1/4	32
$Z = \mathbf{c}_B\mathbf{x}_B = 960$		$\Delta_j = c_j - c_B Y_j$	0	5 (max)	0	-10	

Third Simplex Table

B	$\mathbf{c}_B$	c_j	40	35	0	0	Min. ratio
		$\mathbf{x}_B$	$Y_1(\beta_2)$	$Y_2(\beta_1)$	Y_3	Y_4	
Y_2	35	8	0	1	2/3	-1/3	
Y_1	40	18	1	0	-1/2	1/2	
$Z = \mathbf{c}_B \mathbf{x}_B = 1000$		$\Delta_j = c_j - c_B Y_j$	0	0	-10/3	-25/3	

- Since all $\Delta_j \leq 0$, the solution is Optimal values:

$$x_1 = 18, x_2 = 8$$

- Maximum value:

$$Z_{max} = 1000.$$

Conclusion

- The optimal solution is **unique.**
- Both slack variables s_3, s_4 are non-basic
- Simplex method converges in **two iterations**

Example 2

Solve by simplex method the following LPP

Minimize $Z = x_2 - 3x_3 + 2x_5$

Subject to the constraints

$$\begin{aligned} 3x_2 - x_3 + 2x_5 &\leq 7, \\ -2x_2 + 4x_3 &\leq 12, \\ -4x_2 + 3x_3 + 8x_5 &\leq 10, \\ x_2,\ x_3,\ x_5 &\geq 0. \end{aligned}$$

$$\text{Max } Z' = -Z = 0 \cdot x_1 - x_2 + 3x_3 + 0 \cdot x_4 - 2x_5 + 0 \cdot x_6$$

s.t.

$$\begin{aligned} x_1 + 3x_2 - x_3 + 0x_4 + 2x_5 + 0x_6 &= 7 \\ 0x_1 - 2x_2 + 4x_3 + x_4 + 0x_5 + 0x_6 &= 12 \\ 0x_1 - 4x_2 + 3x_3 + 0x_4 + 8x_5 + x_6 &= 10 \\ x_1, x_2, \dots, x_6 &\geq 0 \end{aligned}$$

Taking $x_2 = 0, x_3 = 0, x_5 = 0$, we get $x_1 = 7, x_4 = 12, x_6 = 10$, which is the initial B.F.S.

B	$\mathbf{c}_B$	c_j	0	-1	3	0	-2	0	**Min. ratio**
		$\mathbf{x}_B$	Y_1	Y_2	Y_3	Y_4	Y_5	T_6	$\mathbf{x}_B/Y_3$
Y_1	0	7	1	3	-1	0	2	0	Neg.
Y_4	0	12	0	-2	4	1	0	0	3 (min) →
Y_6	0	10	0	-4	3	0	8	1	10/3
$Z' = \mathbf{c}_B\mathbf{x}_B = 0$		Δ_j	0	-1	3 ↑	0 ↓	-2	0	x_B/Y_2
Y_1	0	10	1	5/2	0	1/4	2	0	4 (min) →
Y_3	3	3.	0	-1/2	1	1/4	0	0	Neg.
Y_6	0	1	0	-5/2	0	-3/4	8	1	Neg.
$Z' = 9$		Δ_j	0 ↓	1/2 ↑	0	-3/4	-2	0	
Y_2	-1	4	2/5	1	0	1/10	4/5	0	
Y_3	3	5	- 1/5	0	1	3/10	2/5	0	
$\boldsymbol{Y_6}$	0	11	1	0	0	$-1/2$	10	1	
$Z' = 11$		Δ_j	-1/5	0	0	-4/5	-12/5	0	

Hence the optimal solution is

$$x_2 = 4,\ x_3 = 5, x_5 = 0 \text{ and Min } Z = -Z' = -11$$

Example 3

Solve by simplex method the following LPP

Maximize $Z = x_1 + 2x_2 + 3x_3$

Subject to the constraints

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &\leq 10, \\ x_1 + x_2 &\leq 5, \\ x_1 &\leq 1, \\ x_1,\ x_2,\ x_3 &\geq 0. \end{aligned}$$

Maximize $Z = x_1 + 2x_2 + 3x_3 + 0s_4 + 0s_5 + 0s_6$

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + s_4 &= 10, \\ x_1 + x_2 + s_5 &= 5, \\ x_1 + s_6 &= 1, \\ x_1,\ x_2,\ x_3, s_4,\ s_5,\ s_6 &\geq 0. \end{aligned}$$

B	$\mathbf{c}_B$	c_j	0	-1	3	0	-2	0	Min. ratio	
		$\mathbf{x}_B$	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	$\mathbf{x}_B/Y_3$	
Y_4	0	10	1	2	3	1	0	0	10/3 →	
Y_5	0	5	1	1	0	0	1	0	Inf	
Y_6	0	1	1	0	0	0	0	1	Inf	
$Z' = \mathbf{c}_B\mathbf{x}_B = 0$		Δ_j	1	2	3 ↑	0	0	0	$\frac{\mathbf{x}_B}{Y_2}$	$\frac{\mathbf{x}_B}{Y_1}$
Y_3	3	10/3	1/3	2/3	1	1/3	0	0	5	10
Y_5	0	5	1	1	0	0	1	0	5 →	5
Y_6	0	1	1	0	0	0	0	1	Inf.	1 →
$Z' = 10$		Δ_j	0 ↑ 2	0 ↑ 1	0	-1	0	0		

Max Value Z = 10

Optimal Solution: $x_1 = 0, x_2 = 0, x_3 = 10/3$

Alternating Solution I:

B	$\mathbf{c}_B$	c_j	**0**	**-1**	**3**	**0**	**-2**	**0**	**Min. ratio**
		$\mathbf{x}_B$	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	
Y_3	3	0	-1/3	0	1	1/3	-2/3	0	
Y_2	2	5	1	1	0	0	1	0	
Y_6	0	1	1	0	0	0	0	1	
$Z = 10$		Δ_j	0	0	0	-1	0	0	

Max Value Z=10; Optimal Solution: $x_1 = 0, x_2 = 5, x_3 = 0$

Alternating Solution II:

B	$\mathbf{c}_B$	c_j	**0**	**-1**	**3**	**0**	**-2**	**0**	**Min. ratio**
		$\mathbf{x}_B$	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	
Y_3	3	3	0	2/3	1	1/3	0	-1/3	9/2
Y_5	0	4	0	1	0	0	1	-1	4 →
Y_1	1	1	1	0	0	0	0	1	Inf
$Z = 10$		Δ_j	0	0 ↑	0	-1	0	0	

Max Value Z=10; Optimal Solution: $x_1 = 1, x_2 = 0, x_3 = 3$

Alternating Solution III:

B	$\mathbf{c}_B$	c_j	**0**	**-1**	**3**	**0**	**-2**	**0**	**Min. ratio**
		$\mathbf{x}_B$	Y_1	Y_2	Y_3	Y_4	Y_5	T_6	
Y_3	3	1/3	0	0	1	1/3	-2/3	1/3	
Y_2	2	4	0	1	0	0	1	-1	
Y_1	1	1	1	0	0	0	0	1	
$Z = 10$		Δ_j	0	0	0	-1	-1	0	

Max Value Z=10; Optimal Solution: $x_1 = 1, x_2 = 4, x_3 = 1/3$

Artificial Variables Technique

The LPP involving artificial variables can be solved by two methods :

1. Big M-Method (Carners M-Method)
2. Two Phase Method

Big-M Method

- Assign a **very large negative penalty** $(-\mathrm{M}, \mathrm{M} > 0)$ to each artificial variable.
- **Modified Objective Function:**

$$Z = cx + 0 \cdot x_{slack} + 0 \cdot x_{surplus} - M \cdot x_{artificial}$$

- Penalty forces artificial variables to **leave the basis.**
- A solution with **no artificial variable in the basis** is **feasible.**

Example

Apply Big-M Method to solve the following LPP

$$\text{Max. } Z = 2x_1 + 4x_2$$

subject to

$$\begin{aligned} 2x_1 + x_2 &\leq 18 \\ 3x_1 + 2x_2 &\geq 30 \\ x_1 + 2x_2 &= 26 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Introducing the necessary slack variable x_3, surplus variable x_4 and artificial variables, x_{a_1}, x_{a_2} and assigning large negative costs -M to artificial variables, the problem reduces to the form :

$$Max\ Z = 2x_1 + 4x_2 + 0x_3 + 0x_4 - M\,x_{a_1} - Mx_{a_2}$$

subject to

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 18 \\ 3x_1 + 2x_2 - x_4 + x_{a_1} &= 30 \\ x_1 + 2x_2 + x_{a_2} &= 26 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

- Initial BFS:

$x_1 = 0, x_2 = 0, x_4 = 0$ implies
$x_3 = 18,\quad x_{a_1} = 30,\quad x_{a_2} = 26,$

B	c_B	c_j	2	4	0	0	$-M$	$-M$	Min. ratio x_B/Y_2
		x_B	Y_1	Y_2	Y_3	Y_4	A_1	A_2	
Y_3	0	18	2	1	1	0	0	0	18
A_1	$-M$	30	3	2	0	-1	1	0	15
A_2	$-M$	26	1	[2]	0	0	0	1	13(min) →
$Z = c_B x_B = -56M$		Δ_j	$2+4M$	$4+4M$ ↑	0	$-M$	0	0 ↓	x_B/Y_1
Y_3	0	5	3/2	0	1	0	0		10/3
A_1	$-M$	4	[2]	0	0	-1	1		2(min) →
Y_2	4	13	1/2	1	0	0	0		26
$Z = 52 - 4M$		Δ_j	$2M$ ↑	0	0	$-M$	0 ↓		
Y_3	0	2	0	0	1	3/4			
Y_1	2	2	1	0	0	-1/2			
Y_2	4	12	0	1	0	1/4			
$Z = 52$		Δ_j	0	0	0	0			

Optimal solution is $x_1 = 2, x_2 = 12$ and Max. $Z = 52$.

Example

Apply Big-M Method to solve the following LPP

$$\text{Max. } Z = x_1 + 2x_2 + 3x_3 - x_4$$

subject to

$$\begin{aligned} x_1 + 2x_2 + 3x_3 \quad &= 15 \\ 2x_1 + x_2 + 5x_3 \quad &= 20 \\ x_1 + 2x_2 + x_3 + x_4 \quad &= 10 \\ x_1, x_2, x_3, x_4 \quad &\geq 0 \end{aligned}$$

- To obtain a unit matrix of order 3, two additional unit vectors are required.
- One unit vector is already provided by the coefficients of x_4.
- Introduce artificial variables: x_{a_1} in the first constraint and x_{a_2} in the second constraint.
- Assign a large negative cost $-M$ to each artificial variable.

The problem is then converted into:

$$Max\ Z = x_1 + 2x_2 + 3x_3 - x_4 - M\, x_{a_1} - Mx_{a_2}$$

subject to

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + x_{a_1} \quad &= 15 \\ 2x_1 + x_2 + 5x_3 + x_{a_2} \quad &= 20 \\ x_1 + 2x_2 + x_3 + x_4 \quad &= 10 \\ x_1, x_2, x_3, x_4 \quad &\geq 0 \end{aligned}$$

- Initial BFS:

$x_1 = 0, x_2 = 0, x_3 = 0$ implies

$$x_{a_1} = 15, x_{a_2} = 20, x_4 = 10$$

B	$\mathbf{c}_B$	c_j	2	4	0	0	$-M$	$-M$	Min. ratio x_B/Y_2
		$\mathbf{x}_B$	Y_1	Y_2	Y_3	Y_4	A_1	A_2	
Y_3	0	18	2	1	1	0	0	0	18
A_1	$-M$	30	3	2	0	-1	1	0	15
A_2	$-M$	26	1	[2]	0	0	0	1	13(min) →
$Z = \mathbf{c}_B\mathbf{x}_B = -56M$		Δ_j	$2+4M$	$4+4M$ ↑	0	$-M$	0	0 ↓	x_B/Y_1
Y_3	0	5	3/2	0	1	0	0		10/3
A_1	$-M$	4	[2]	0	0	-1	1		2(min) →
Y_2	4	13	1/2	1	0	0	0		26
$Z = 52 - 4M$		Δ_j	$2M$ ↑	0	0	$-M$	0 ↓		
Y_3	0	2	0	0	1	3/4			
Y_1	2	2	1	0	0	-1/2			
Y_2	4	12	0	1	0	1/4			
$Z = 52$		Δ_j	0	0	0	0			

Optimal solution is $x_1 = 2, x_2 = 12$ and Max. $Z = 52$.

Example

Apply Big-M Method to solve the following LPP

$$\text{Max. } Z = -x_1 - x_2$$

subject to

$$\begin{aligned} 3x_1 + 2x_2 &\geq 30 \\ -2x_1 + 3x_2 &\leq -30 \\ x_1 + x_2 &\leq 5 \\ x_1, x_2 &\geq 0 \end{aligned}$$

$$Max\ Z = -x_1 - x_2 + 0x_3 + 0x_4 + 0x_5 - M\, x_{a_1} - Mx_{a_2}$$

subject to

$$\begin{aligned} 3x_1 + 2x_2 - x_3 + x_{a_1} &= 30 \\ 2x_1 - 3x_2 - x_4 + x_{a_2} &= 30 \\ x_1 + x_2 + x_5 &= 5 \\ x_1, x_2, x_3, x_4, x_5, x_{a_1}, x_{a_2} &\geq 0 \end{aligned}$$

- Initial BFS:

$x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0$ implies
$x_5 = 5,\quad A_1 = x_{a_1} = 30,\quad A_2 = x_{a_2} = 30,$

B	c_B	c_j	-1	-1	0	0	0	$-M$	$-M$	Min. ratio
		x_B	Y_1	Y_2	Y_3	Y_4	Y_5	A_1	A_2	x_B/Y_1
A_1	$-M$	30	3	2	-1	0	0	1	0	10
A_2	$-M$	30	2	-3	0	-1	0	0	1	15
Y_5	0	5	[1]	1	0	0	1	0	0	5 (min) →
$Z = c_B x_B = -60M$		Δ_j	$5M - 1$ ↑	$-M$	$-M$	$-M$	0 ↓	0	0	
A_1	$-M$	15	0	-1	-1	0	-3	1	0	
A_2	$-M$	20	0	-5	0	-1	-2	0	1	
Y_1	-1	5	1	1	0	0	1	0	0	
$Z = c_B x_B = -35M - 5$		Δ_j	0	$-6M$	$-M$	$-M$	$-5M$	0	0	

Since all $\Delta_j \leq 0$, therefore the solution obtained is optimal. But the artificial variables appear in the basis at positive levels which implies that the given L.P.P. has no feasible solution.

Example

Apply Big-M Method to solve the following LPP

$$\text{Max. } Z = 10x_1 + 20x_2$$

subject to

$$2x_1 + 4x_2 \geq 16$$
$$x_1 + 4x_2 \geq 15$$
$$x_1, x_2 \geq 0$$

$$Max\ Z = -x_1 - x_2 + 0x_3 + 0x_4 - M\,x_{a_1} - Mx_{a_2}$$

subject to

$$2x_1 + 4x_2 - x_3 + x_{a_1} = 16$$
$$x_1 + 5x_2 - x_4 + x_{a_2} = 15$$
$$x_1, x_2, x_3, x_4, x_{a_1}, x_{a_2} \geq 0$$

- Initial BFS:

$x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0$ implies
$A_1 = x_{a_1} = 16$, $A_2 = x_{a_2} = 15$,

B	c_B	c_j	10	20	0	0	$-M$	$-M$	Min. ratio x_B/Y_2
		$\mathbf{x}_B$	Y_1	Y_2	Y_3	Y_4	A_1	A_2	
A_1	$-M$	16	2	4	-1	0	1	0	16/4
A_2	$-M$	15	1	[5]	0	-1	0	1	15/5(min) →
$Z = \mathbf{c}_B\mathbf{x}_B = -31M$		Δ_j	$3M+10$	$9M+20$ ↑	$-M$	$-M$	0	0 ↓	x_B/Y_1
A_1	$-M$	4	[6/5]	0	-1	-4/5	1		10/3 (min) →
Y_2	20	3	1/5	1	0	-1/5	0		15
$Z = 60 - 4M$		Δ_j	$6\,(M+5)/5$ ↑	0	$-M$	$4\,(M+5)/5$	0 ↓		x_B/Y_3
Y_1	10	10/3	1	0	-5/6	2/3			Neg.
Y_2	20	7/3	0	1	[1/6]	-1/3			14 (min) →
$Z = 80$		Δ_j	0	0 ↓	5 ↑	0			
Y_1	10	15	1	5	0	-1			
Y_3	0	14	0	6	1	-2			
Z = 150		Δ_j	0	-30	0	10 ↑			

x_4 is the incoming vector but we cannot find outgoing vector because all the elements in this column are negative. Therefore, the solution is **unbounded**.

Example

(LPP with unrestricted variables)

$$\text{Max. } Z = 2x_1 + 3x_2$$

subject to

$$-x_1 + 2x_2 \le 4$$
$$x_1 + x_2 \le 6$$
$$x_1 + 3x_2 \le 9,$$

x_1, x_2 are unrestricted.

STEP I

- Simplex method requires **non-negative variables.**
- Since x_1, x_2 are unrestricted, use the transformation

$$x_1 = x_1' - x_1'', \qquad x_2 = x_2' - x_2'',$$

$$x_1', x_1'', x_2', x_2'' \geq 0$$

- The given problem becomes

$$\text{Max. } Z = 2(x_1' - x_1'') + 3(x_2' - x_2'')$$

$$-(x_1' - x_1'') + 2(x_2' - x_2'') \leq 4$$
$$(x_1' - x_1'') + (x_2' - x_2'') \leq 6$$
$$(x_1' - x_1'') + 3(x_2' - x_2'') \leq 9,$$
$$x_1', x_2', x_1'', x_2'' \geq 0$$

STEP II

- Introduce slack variables x_3, x_4, x_5 to convert inequalities into equations as

$$\text{Max. } Z = 2x_1' - 2x_1'' + 3x_2 - 3x_2'' + 0 \cdot x_3 + 0 \cdot x_4 + 0 \cdot x_5$$
$$-x_1' + x_1'' + 2x_2' - 2x_2'' + x_3 = 4$$
$$x_1' - x_1'' + x_2' - x_2'' + x_4 = 6$$
$$x_1' - x_1'' + 3x_2' - 3x_2'' + x_5 = 9$$

- Standard form obtained with all variables non-negative
- **Starting BFS** is obtained by setting

$$x_1' = x_1'' = x_2' = x_2'' = 0$$

Then

$$x_3 = 4, x_4 = 6, \ x_5 = 9$$

gives the **initial basic feasible solution**

B	c_B	c_j	**2**	**-2**	**3**	**-3**	**0**	**0**	**0**	**Min. ratio**
		x_B	$Y_1{}'$	$Y_1{}''$	$Y_2{}'$	Y_2''	Y_3	Y_4	Y_5	$x_B/Y_2{}'$
Y_3	0	4	-1	1	[2]	-2	1	0	0	2(min) →
Y_4	0	6	1	-1	1	-1	0	1	0	6
Y_5	0	9	1	-1	3	-3	0	0	1	3
$Z = c_B\mathbf{x}_B$ = **0**		Δ_j	2	-2	3 ↑	-3	0 ↓	0	0	$\mathbf{x}_B/Y_1'$
Y_2'	3	2	-1/2	1/2	1	-1	1/2	0	0	Neg.
Y_4	0	4	3/2	-3/2	0	0	-1/2	1	0	8/3
Y_5	0	3	[5/2]	$-5/2$	0	0	-3/2	0	1	6/5 (min) →
$Z = c_B x_B$ = **6**		Δ_j	7/2 ↑	-7/2	0	0	-3/2	0	0 ↓	$\mathbf{x}_B/Y_3$
Y_2'	3	13/5	0	0	1	-1	1/5	0	1/5	13
Y_4	0	11/5	0	0	0	0	[2/5]	1	-3/5	11/2 (min) →
Y_1'	2	6/5	1	-1	0	0	-3/5	0	2/5	Neg.
$Z = c_B x_B$ =**51/5**		Δ_j	0	0	0	0	3/5 ↑	0 ↓	-7/5	
Y_2'	3	3/2	0	0	1	-1	0	-1/2	1/2	
Y_3	0	11/2	0	0	0	0	1	5/2	-3/2	
Y_1'	2	9/2	1	-1	0	0	0	3/2	-1/2	
$Z = c_B x_B$ =**27/2**		Δ_j	0	0	0	0	0	-3/2	-1/2	

Since all $\Delta_j \leq 0$, therefore the solution is optimal, and the optimal solution is $x_1' = 9/2, x_1'' = 0, x_2' = 3/2, x_2'' = 0,$
$\Rightarrow x_1 = x_1' - x_1'' = 9/2, x_2 = x_2' - x_2'' = 3/2$, Max. $Z = 27/2$.

Conditions for the Occurrence of Degeneracy in a LPP

Degeneracy in a Linear Programming Problem occurs when **one or more basic variables take a value of zero** at a basic feasible solution.

When Degeneracy Appears?

- The degeneracy may appear in a LPP at the very first stage when some b_i is zero or at any iteration when the value of some variable in the basis becomes zero.
- If none of the component of **b** is zero at any iteration and the choice of the outgoing vector at the same iteration is not unique then the next iteration is bound to be degenerate.

- Degeneracy can cause the simplex iterations to cycle indefinitely, thus never terminating the algorithm
- It reveals the possibility of at least one redundant constraint.

Resolution of Degeneracy

Given:

a_k is the vector entering the basis and

$$\min_i \left\{ \frac{y_{B_i}}{y_{ik}} \mid y_{ik} > 0 \right\}$$

is **not unique.**

Step I: Renumber Columns

- Renumber the columns of the simplex tableau starting with the columns in the basis, say, $\bar{Y}_1, \bar{Y}_2, \ldots$
- Let $\bar{Y}_t$ be the renumbered entering vector such that $Y_k = \bar{Y}_t$

Step II: Identify Tied Indices

- If $\min_i \left\{ \frac{y_{B_i}}{y_{ik}} \mid y_{ik} > 0 \right\}$ occurs at:
  $$i = i_1, i_2, \ldots, i_s$$
- Define the index set:
  $$I_1 = \{i_1, i_2, \ldots, i_s\}$$

Compute

$$\min_{i \in I_1} \left\{ \frac{\bar{y}_{i1}}{\bar{y}_{it}}, \bar{y}_{it} > 0 \right\}.$$

Decision:

- If **unique** value → Remove the corresponding vector from the basis.
- If **not unique** → Proceed to Step 3.

Step III

Compute

$$\min_{i \in I_2} \left\{ \frac{\bar{y}_{i2}}{\bar{y}_{it}}, \bar{y}_{it} > 0 \right\}$$

where

$$I_2 \subset I_1$$

contains only those indices tied in Step II.

Decision:

If **unique** value → Remove corresponding vector from the basis

If **not unique** value → Continue to Step IV.

Step IV

Compute

$$\min_{i \in I_3} \left\{ \frac{\bar{y}_{i3}}{\bar{y}_{it}}, \bar{y}_{it} > 0 \right\}$$

where

$$I_3 \subset I_2 \subset I_1$$

for which there is a tie in Step III.

Continuing in this way we shall get a unique minimum value, i.e., the unique vector to be deleted from this basis.

Example

Solve the following LPP by simplex method

$$\text{Max. } Z = 2x_1 + x_2$$

Subject to

$$4x_1 + 3x_2 \leq 12$$
$$4x_1 + x_2 \leq 8$$
$$4x_1 - x_2 \leq 8$$
$$x_1, x_2 \geq 0$$

Introducing the slack variables x_3, x_4 and x_5, the given L.P.P. can be written as

$$\begin{aligned} \text{Max.} \quad & Z = 2x_1 + x_2 \\ \text{s.t.} \quad & 4x_1 + 3x_2 + x_3 = 12 \\ & 4x_1 + x_2 + x_4 = 8 \\ & 4x_1 - x_2 + x_5 = 8 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

Taking $x_1 = 0, x_2 = 0$, we get

$$x_3 = 12,\ x_4 = 8,\ x_5 = 8,$$

which is the starting BFS.

B	$\mathbf{c}_B$	c_j	2	1	0	0	0	Min. Ratio
		x_B	$\bar{Y}_4$ Y_1	$\bar{Y}_5$ Y_2	$\bar{Y}_1$ Y_3	$\bar{Y}_2$ Y_4	$\bar{Y}_3$ Y_5	$\mathbf{x}_B/Y_1$
Y_3	0	12	4	3	1	0	0	3
Y_4	0	8	4 $(\bar{y}_{24})$	1	0 $(\bar{y}_{21})$	1 $(\bar{y}_{22})$	0	2
Y_5	0	8	**4** $(\bar{y}_{34})$	-1	0 $(\bar{y}_{31})$	0 $(\bar{y}_{32})$	1	2 →
$Z = \mathbf{c}_B \cdot \mathbf{x}_B = 0$		Δ_j	2 ↑	1	0	0	0 ↓	

Since **positive** Δ_j exist, the solution is **not optimal**

$$\max(\Delta_j) = 2 = \Delta_1$$

$\Rightarrow$ **Entering (Incoming) Vector** $= \alpha_1 = Y_1$

By the **minimum ratio rule**, the minimum occurs for:

$$i = 2 \text{ and } i = 3$$

$\Rightarrow$ Minimum is not unique

$\Rightarrow$ **Problem is degenerate**

Step I: Reorder columns so identity matrix appears first:

$\bar{Y}_1 = Y_3, \bar{Y}_2 = Y_4, \bar{Y}_3 = Y_5, \bar{Y}_4 = Y_1$ and $\bar{Y}_5 = Y_2$.

Step II: Since minimum ratio occurs for $i = 2$ and 3,

$$I_1 = \{2,3\}.$$

Incoming vector:

$$Y_1 = \bar{Y}_4 \Rightarrow t = 4$$

$$\therefore \min_{i \in I_1} \left\{ \frac{\bar{y}_{i1}}{\bar{y}_{it}}, \bar{y}_{it} > 0 \right\} = \min_{i \in I_1} \left\{ \frac{\bar{y}_{i1}}{\bar{y}_{i4}}, \bar{y}_{i4} > 0 \right\} = \min \left\{ \frac{\bar{y}_{21}}{\bar{y}_{24}}, \frac{\bar{y}_{31}}{\bar{y}_{34}} \right\} = \min \left\{ \frac{0}{4}, \frac{0}{4} \right\} = \{0,0\}$$

Minimum is not unique and there is a tie at $i = 2,3$. Thus, $I_2 = \{2,3\}$.

Step III: $\min_{i \in I_2} \left\{ \frac{\bar{y}_{i2}}{\bar{y}_{it}}, \bar{y}_{it} > 0 \right\} = \min_{i \in I_2} \left\{ \frac{\bar{y}_{22}}{\bar{y}_{24}}, \frac{\bar{y}_{32}}{\bar{y}_{34}} \right\} = \min \left\{ \frac{1}{4}, \frac{0}{4} \right\} = \frac{0}{4} = \frac{\bar{y}_{32}}{\bar{y}_{34}}.$

Minimum occurs at $i = 3$.

Conclusion:

Outgoing Vector$= \bar{Y}_3 = Y_5$

Key Element $= \bar{y}_{34} = 4$

B	c_B	c_j	2	1	0	0	0	Min. Ratio
		$\mathbf{x}_B$	Y_1	Y_2	Y_3	Y_4	Y_5	$\mathbf{x}_B/Y_2$
Y_3	0	4	0	4	1	0	-1	1
Y_4	0	0	0	2	0	1	-1	0 (Mini)
Y_1	2	2	1	-1/4	0	0	1/4	Neg.
$Z = c_B \cdot x_B = 4$		Δ_j	0	3/2	0	0	-1/2	$\mathbf{x}_B/Y_5$
Y_3	0	4	0	0	1	-2	1	4 (Mini)
Y_2	1	0	0	1	0	1/2	-1/2	Neg.
Y_1	2	2	1	0	0	1/8	1/8	16
$Z = c_B \cdot x_B = 4$		Δ_j	0	0	0	-3/4	1/4	
Y_5	0	4	0	0	1	-2	1	
Y_2	1	2	0	1	1/2	1/2	0	
Y_1	2	3/2	1	0	-1/8	3/8	0	
$Z = c_B \cdot x_B = 5$		Δ_j	0	0	-1/4	-1/2	0	

Optimal solution is: $x_1 = 3/2, x_2 = 2$ and Max. $Z = 5$

Assignment.

Using simplex method to solve the following LPP:

$$\text{Max. } 6x_1 + 3x_2 + 5$$

subject to

$$x_1 + 3x_2 \leq 9$$
$$x_1 + x_2 \geq 5$$
$$2x_1 + x_2 \leq 8$$
$$x_1, x_2 \geq 0$$