OPERATIONS RESEARCH

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Duality in Linear Programming

A fundamental discovery in optimization theory that reveals the deep connection between primal and dual problems.

Primal: the original problem
Dual: the relate problem

The Core Concept

Some times, it's easier to solve the dual problem than the primal, making duality a powerful computational tool.

Two Problems, One Solution

Every linear programming problem has a corresponding dual problem. They are mathematical replicas of each other.

Opposite Directions

If the primal is a maximization problem, the dual will be minimization, and vice versa.

Linked Solutions

Knowing the optimal solution to one problem allows us to easily find the optimal solution to the other.

The Diet Problem

Two foods provide vitamins A and B. We need to meet minimum daily requirements at minimum cost.

Vitamin	Food		Minimum daily requirement
	A_1	A_2	
A	6	9	60
B	4	13	108
Cost (per unit)	Rs 12	Rs 18	

Formulating the Primal

Let x_1 and x_2 be the number of units of foods A_1 and A_2 to be purchased respectively.

Objective Function:

Minimize $Z_x = 12x_1 + 18x_2$

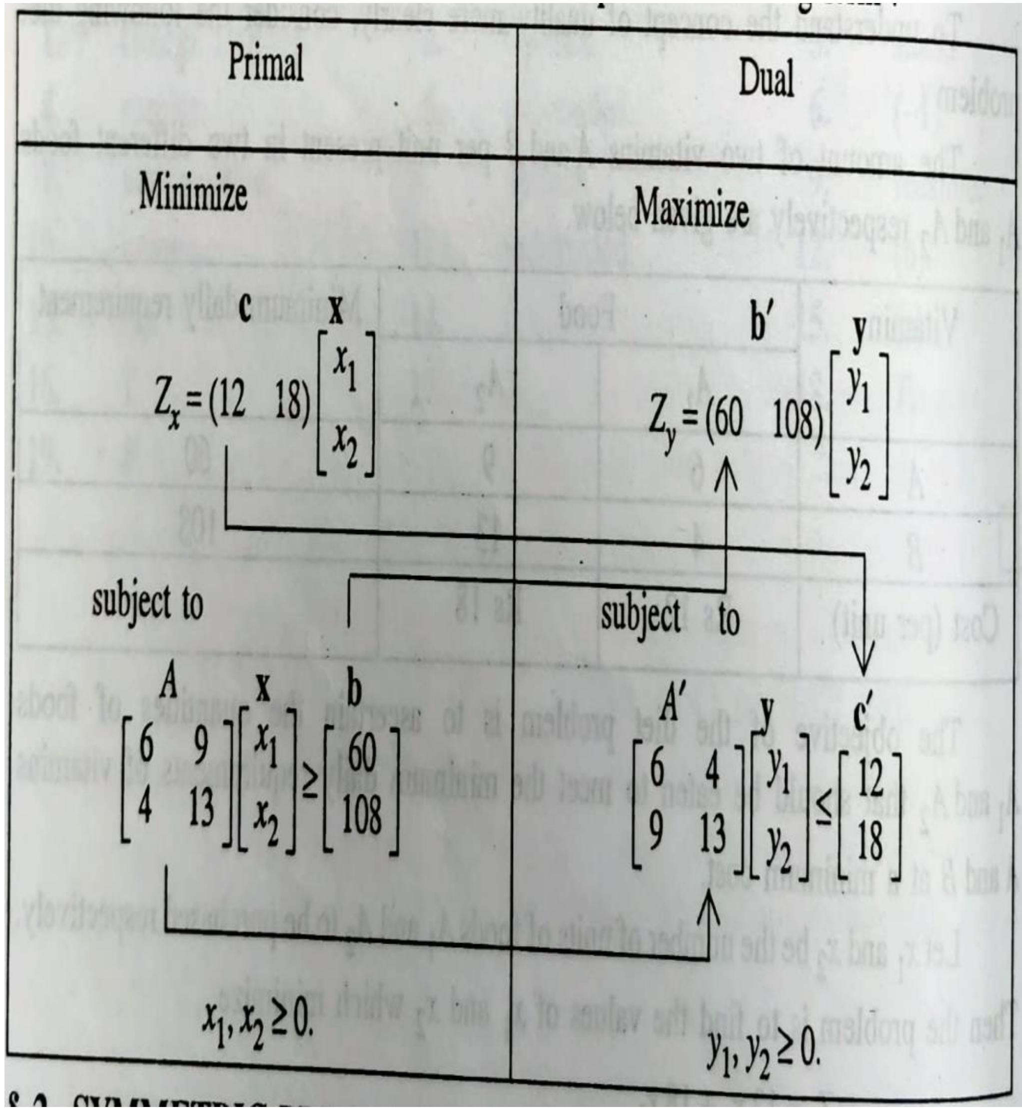
Constraints:

$$6x_1 + 9x_2 \geq 60,$$
$$4x_1 + 13x_2 \geq 108,$$
$$x_1, x_2 \geq 0.$$

The Dual Perspective

Wholesale Dealer's Problem

- A dealer sells vitamins A and B. He must set maximum per-unit prices such that the resulting prices of foods A_1 and A_2 don't exceed their market values.
- The foods have market value only because of their vitamin content.



Formulating the Dual

Variables: Let y_1 and y_2 are prices per unit of vitamins A and B, respectively.

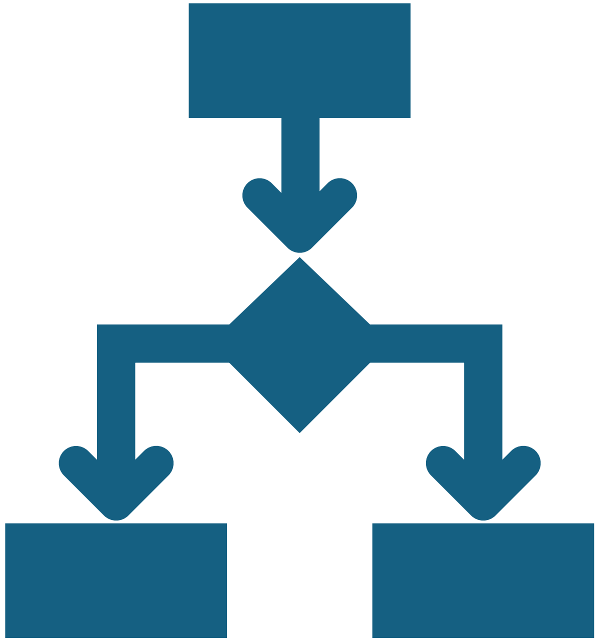
Objective Function:

$$\text{Maximize } Z_y = 60y_1 + 108y_2$$

Constraints:

$$6y_1 + 4y_2 \leq 12$$
$$9y_1 + 13y_2 \leq 18,$$
$$y_1, y_2 \geq 0.$$

Primal-Dual Relationship



- **Problem Direction**

  Primal minimizes, dual maximizes (or vice versa).
- **Value Swap**

  Constraint values become objective coefficients and vice versa.
- **Matrix Transpose**

  Dual coefficient matrix is transpose of primal coefficient matrix.
- **Inequality Reversal**

  Direction of inequalities reverses between primal and dual.

Symmetric Primal-Dual Problems

Primal L.P.P.

Find the variables $x_1, x_2, \dots, x_n$ which **maximize**

$$Z_x = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

subject to

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2$$

$$\dots$$

$$\dots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m$$

and

$$x_1, x_2, \dots, x_n \ge 0$$

the signs of the parameters a, b, c are arbitrary.

Dual problem

Find the variables $w_1, w_2, \dots, w_m$ which **minimize**

$$Z_w = b_1w_1 + b_2w_2 + \cdots + b_mw_m$$

subject to

$$a_{11}w_1 + a_{21}w_2 + \cdots + a_{m1}w_m \ge c_1$$

$$a_{12}w_1 + a_{22}w_2 + \cdots + a_{m2}w_m \ge c_2$$

$$\dots$$

$$\dots$$

$$a_{1n}w_1 + a_{2n}w_2 + \cdots + a_{mn}w_m \ge c_n$$

and

$$w_1, w_2, \dots, w_n \ge 0$$

Matrix Form of Symmetric Primal-Dual Problem

Primal Problem.

Find a column vector $\mathbf{x} \in R^n$ which maximizes

$$Z_x = \mathbf{cx}, \mathbf{c} \in R^n$$

subject to

$$A\mathbf{x} \le \mathbf{b}, \mathbf{b} \in R^m,$$

$x \ge 0$ and A is an $m \times n$ real matrix.

Dual Problem.

Find a column vector w $\in R^m$ which minimizes

$$Z_w = \mathbf{b}'\mathbf{w}$$

subject to

$$A'\mathbf{w} \ge \mathbf{c}'$$

$\mathbf{w} \ge 0, A', \mathbf{b}', \mathbf{c}'$ are the transposes of A, $\mathbf{b}$ and $\mathbf{c}$ respectively.

Example

Consider the symmetric primal problem

$$\text{Max. } Z_x = 5x_1 + 9x_2$$

subject to

$$\begin{aligned} x_1 &\leq 6 \\ x_1 + x_2 &\leq 13 \\ x_2 &\leq 8 \\ x_1, x_2 &\geq 0 \end{aligned}$$

The corresponding dual problem is

$$\text{Min. } Z_w = 6w_1 + 13w_2 + 8w_3$$

subject to

$$\begin{aligned} w_1 + w_2 &\geq 5 \\ w_2 + w_3 &\geq 9 \\ w_1, w_2, w_3 &\geq 0 \end{aligned}$$

Unsymmetric Primal-Dual Problems

Primal Problem.

Find a column vector $\mathbf{x} \in R^n$ which maximizes

$$Z_x = \mathbf{cx}, \mathbf{c} \in R^n$$

subject to $A\mathbf{x} = \mathbf{b}, \mathbf{b} \in R^m$,
$\mathbf{x} \geq 0$ and A is an $m \times n$ real matrix.

Dual Problem.

Find a column vector w $\in R^m$ which minimizes

$$Z_w = \mathbf{b}'\mathbf{w}$$

subject to $A'\mathbf{w} \geq \mathbf{c}'$.
In this case the dual variables are unrestricted in sign.

Key Takeaway: The dual variables corresponding to primal equality constraints must be unrestricted in sign and those associated with primal inequalities must be non-negative.

Dual of an LPP with Mixed Restrictions

1. Equation Handling:

Replace each equation in the primal problem with two inequalities going in opposite directions (≤ and ≥).

Example: the equation $2x_1 + 5x_2 = 9$ is replaced by

$2x_1 + 5x_2 \leq 9$ and $2x_1 + 5x_2 \geq 9$.

2. Sign Adjustment:

For maximization problems, ensure all constraints have ≤ sign, and for minimization problems, all constraints have ≥ sign. (Multiply by -1 if needed).

3. Unrestricted Variables:

Replace unrestricted variables with difference of two non-negative variables.

4. Find the Dual:

Now find the dual problem as usual procedure.

Standard Primal Form: An LPP is said to be in standard primal form if

(a) for a maximization problem all the constraints have ≤ sign.

(b) for a minimization problem all the constraints have ≥ sign.

Example 1.

Find the dual of the following LPP

Min. $Z = 10x_1 + 20x_2$

subject to

$$
\begin{aligned}
3x_1 + 2x_2 &\geq 18 \\
x_1 + 3x_2 &\geq 8 \\
2x_1 - x_2 &\leq 6, \\
x_1, x_2 &\geq 0
\end{aligned}
$$

Standard primal form of given LPP:

$$\text{Min. } Z = 10x_1 + 20x_2$$

subject to

$$
\begin{aligned}
3x_1 + 2x_2 &\geq 18 \\
x_1 + 3x_2 &\geq 8 \\
-2x_1 + x_2 &\geq -6 \\
x_1, x_2 &\geq 0
\end{aligned}
$$

The matrix form of the primal problem is

Min. $Z = 10x_1 + 20x_2 = (10,20)[x_1, x_2] = \mathbf{cx}$

s.t.

$$\begin{bmatrix} 3 & 2 \\ 1 & 3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq \begin{bmatrix} 18 \\ 8 \\ -6 \end{bmatrix}$$

$$A\mathbf{x} \geq \mathbf{b}, x_1, x_2 \geq 0.$$

The dual of this problem is

Max. $Z_D = \mathbf{b}'\mathbf{y} = (18,8,-6)[y_1, y_2, y_3]$

s.t.

$$A'\mathbf{y} \leq \mathbf{c}' \text{ or } \begin{bmatrix} 3 & 1 & -2 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \leq \begin{bmatrix} 10 \\ 20 \end{bmatrix}$$

$$\text{or} \quad \begin{bmatrix} 3y_1 + y_2 - 2y_3 \\ 2y_1 + 3y_2 + y_3 \end{bmatrix} \leq \begin{bmatrix} 10 \\ 20 \end{bmatrix}$$

Hence dual of the given problem is

$$\text{Max. } Z_D = 18y_1 + 8y_2 - 6y_3$$

s.t.

$$3y_1 + y_2 - 2y_3 \leq 10,$$
$$2y_1 + 3y_2 + y_3 \leq 20,$$
$$y_1, y_2, y_3 \geq 0.$$

Example 2.

Write the dual of the following problem:

$$\text{Min. } Z = 2x_2 + 5x_3$$

subject to

$$\begin{aligned} x_1 + x_2 \quad\quad\quad &\geq 2 \\ 2x_1 + x_2 + 6x_3 &\leq 6 \\ x_1 - x_2 + 3x_3 &= 4 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

First, we convert the given problem to standard primal form:

$$\text{Min. } Z = 0x_1 + 2x_2 + 5x_3$$

subject to

$$\begin{aligned} x_1 + x_2 \quad\quad\quad &\geq 2 \\ -2x_1 - x_2 - 6x_3 &\geq -6 \\ x_1 - x_2 + 3x_3 &\geq 4 \\ -x_1 + x_2 - 3x_3 &\geq -4 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

The matrix form of the standard primal form is

$$\text{Min. } Z = (0,2,5)[x_1, x_2, x_3] = \mathbf{c}.\mathbf{x}$$

subject to

$$\begin{bmatrix} 1 & 1 & 0 \\ -2 & -1 & -6 \\ 1 & -1 & 3 \\ -1 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \geq \begin{bmatrix} 2 \\ -6 \\ 4 \\ -4 \end{bmatrix} \text{ or } A\mathbf{x} \geq \mathbf{b}, x_1, x_2, x_3 \geq 0$$

Thus, the dual of the given primal is

$$\text{Max. } Z_D = \mathbf{b}'.\mathbf{y} \quad = (2, -6, 4, -4)[y_1, y_2, y_3', y_3'']$$
$$= 2y_1 - 6y_2 + 4(y_3' - y_3'')$$

subject to

$$A'y \leq c' \text{ or } \begin{bmatrix} 1 & -2 & 1 & 1 \\ 1 & -1 & -1 & -1 \\ 0 & -6 & 3 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3' \\ y_3'' \end{bmatrix} \leq \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}$$

$$\text{or } \begin{bmatrix} y_1 - 2y_2 + y_3' - y_3'' \\ y_1 - y_2 - y_3' + y_3'' \\ 0.y_1 - 6y_2 + 3y_3' - 3y_3'' \end{bmatrix} \leq \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}$$

$$y_1, y_2, y_3', y_3'' \geq 0$$

We can further write

$$\text{Max. } Z_D = 2y_1 - 6y_2 - 4(y_3' - y_3'')$$

subject to

$$y_1 - 2y_2 + (y_3' - y_3'') \leq 0$$
$$y_1 - y_2 - (y_3' - y_3'') \leq 2$$
$$-6y_2 + 3(y_3' - y_3'') \leq 5,$$
$$y_1, y_2, y_3', y_3'' \geq 0.$$

Substituting $y_3 = y_3' - y_3''$, **the required dual is**

$$\text{Max. } Z_D = 2y_1 - 6y_2 + 4y_3$$

subject to

$y_1 - 2y_2 + y_3 \leq 0, y_1 - y_2 - y_3 \leq 2, -6y_2 + 3y_3 \leq 5,$

$y_1, y_2 \geq 0$ and y_3 is unrestricted in sign.

Example 3

Write the dual of the following problem:

$$\text{Max. } Z = 3x_1 + 5x_2 + 7x_3$$

subject to

$$x_1 + x_2 + 3x_3 \leq 10$$

$$4x_1 - x_2 + 2x_3 \geq 15$$

$$x_1, x_2 \geq 0, x_3 \text{ is unrestricted.}$$

Substituting $x_3 = x_3' - x_3''$ we get the standard primal form of the given problem as

$$\text{Max. } Z = 3x_1 + 5x_2 + 7(x_3' - x_3'')$$

subject to

$$\begin{aligned} x_1 + x_2 + 3x_3' - 3x_3'' &\leq 10 \\ -4x_1 + x_2 - 2x_3 + 2x_3'' &\leq -15 \\ x_1, x_2, x_3', x_3'' &\geq 0. \end{aligned}$$

The matrix form of the above problem is:

subject to

$$\begin{bmatrix} 1 & 1 & 3 & -3 \\ -4 & 1 & -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3' \\ x_3'' \end{bmatrix} \leq \begin{bmatrix} 10 \\ -15 \end{bmatrix}$$

or $A\mathbf{x} \leq \mathbf{b}$, $x_1, x_2, x_3', x_3'' \geq 0$.

The dual of the given problem is

$$\text{Min. } Z_D = \mathbf{b}'\mathbf{y} = (10, -15)[y_1, y_2] = 10y_1 - 15y_2$$

subject to $A\mathbf{y} \geq \mathbf{c}'$

$$\text{or } \begin{bmatrix} 1 & -4 \\ 1 & 1 \\ 3 & -2 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \geq \begin{bmatrix} 3 \\ 5 \\ 7 \\ -7 \end{bmatrix}$$

$$\text{or } \begin{bmatrix} y_1 - 4y_2 \\ y_1 + y_2 \\ 3y_1 - 2y_2 \\ -3y_1 + 2y_2 \end{bmatrix} \geq \begin{bmatrix} 3 \\ 5 \\ 7 \\ -7 \end{bmatrix}$$

Which is

$$\text{Min. } Z_D = 10y_1 - 15y_2$$

subject to

$$y_1 - 4y_2 \geq 3, \qquad y_1 + y_2 \geq 5,$$

$$3y_1 - 2y_2 \geq 7, \qquad -3y_1 + 2y_2 \geq -7$$

$$y_1, y_2 \geq 0.$$

Hence the required dual problem is

$$\text{Min. } Z_D = 10y_1 - 15y_2$$

subject to

$$y_1 - 4y_2 \geq 3,$$
$$y_1 + y_2 \geq 5,$$
$$3y_1 - 2y_2 = 7,$$
$$y_1, y_2 \geq 0.$$

DUALITY RESULTS

- The dual of a dual of the given primal is the primal itself.

- If $\mathbf{x}$ is any feasible solution to the primal problem

$$Max\ Z_p = \mathbf{cx}, \quad \text{s.t.} \quad A\mathbf{x} \leq \mathbf{b}, \quad \mathbf{x} \geq \mathbf{0}$$

  and $\mathbf{w}$ is any feasible solution to the dual problem

$$Min\ Z_D = \mathbf{b}'\mathbf{w} \quad \text{s.t.} \quad A'\mathbf{w} \geq \mathbf{c}', \quad \mathbf{w} \geq \mathbf{0},$$

  then $\mathbf{cx} \leq \mathbf{b}'\mathbf{w}$, *that is*, $Z_P \leq Z_D$.

- The necessary and sufficient condition for any linear programming problem and its dual to have optimal solution is that both have feasible solutions.

- (Basic Duality theorem.) If $\mathbf{x_0}$ is an optimum solution to the primal, then there exists a feasible solution w_0 to the dual such that

$$c\mathbf{x_0} = \mathbf{b}'w_\mathbf{0}$$

  where $\mathbf{b}'$ is the transpose of $\mathbf{b}$.

- If any of the constraints in the primal is a perfect equality, the corresponding dual variable is unrestricted in sign.

- If any variable of the primal is unrestricted in sign, the corresponding constraint in the dual will be a strict equality.

Fundamental Duality Theorem

1) If either the primal or the dual problem has a finite optimal solution then the other problem also has a finite optimal solution and the optimal values of the objective function in both the problems are the same.

2) If primal (dual) problem has an unbounded optimum solution, the other problem has either no solution at all or an unbounded solution.

Existence Theorems

1. There exists a bounded (finite) optimum solution to an LPP if and only if there exists a feasible solution to both primal and its dual.

2. If there does not exist any feasible solution to the dual (primal) but there exists at least one to the primal (dual), then there does not exist any finite optimum solution to the primal (dual).

3. If there does not exist any finite optimum solution to the primal (dual) then there does not exist any feasible solution to the dual (primal).

	Primal Problem	Dual Problem
1.	Objective function Max. Z_P.	Objective function Min. Z_D.
2	Requirement vector.	Price vector.
3	Coefficient matrix A	Transpose of the coefficient matrix, A' or A^T
4	Constraints with $\leq$ sign.	Constraints with $\geq$ sign.
5	Relation.	Variable.
6	i-th inequality	i-th variable $w_i \geq 0$.
7	i-th constraint an equality.	i-th variable w_i unrestricted in sign.
8	Variable.	Relation.
9	i-th variable $x_i > 0$.	i-th relation a strict inequality.
10	i-th variable x_i unrestricted in sign.	i-th constraint a strict equality.
11	i-th slack variable positive.	i-th variable zero.
12	i-th variable zero.	i-th surplus variable positive.
13	Finite optimal solution.	Finite optimal solution with equal optimal value of objective function.
14	Unbounded solution	No solution or an unbounded solution

Relationship Between Final Simplex Tables of Primal and Dual

From the **final simplex table of the primal problem**, we can directly read the **optimal solution of the dual problem**, and **vice-versa**.

RULES:

- The optimal value of the primal objective function is equal to the optimal value of the dual objective function.

$$\boxed{\max Z_P = \min Z_D}$$

- In the final simplex table of the primal problem, take the values of

$$\Delta_j = c_j - Z_j$$

and change the sign for the slack (or surplus) variables.

The resulting numbers give the optimal values of the corresponding dual variables in the final simplex table of the dual problem.

- If either problem has unbounded solution, then the other will have no feasible solutions.

Why Duality Matters

Computational Advantage:

Sometimes it is easier to solve the dual than the primal problem.

- Compare the primal and dual problems.
- Identify which one has fewer constraints.
- First, solve that problem using the simplex method.
- Then, use the final simplex table to read the solution of the other problem using the duality rules.

Applications:

- Physics

  Used in:
  - Parallel circuit theory
  - Series circuit theory
- Economics

  Applied in:
  - Input–output models
  - Resource allocation systems
- Game Theory

  Used to find optimal strategies. For example:

  If Player B minimizes losses, duality allows us to Convert Player A’s problem into Player B’s problem and vice-versa

Example

Write the dual of the following problem and solve it.

$$\text{Max. } Z = 4x_1 + 2x_2$$

subject to

$$-x_1 - x_2 \leq -3$$
$$-x_1 + x_2 \leq -2,$$
$$x_1, x_2 \geq 0$$

Hence or otherwise write down the solution of the primal.

The given problem is in standard primal form.
Thus, the dual to the given primal is

$$\text{Min. } Z_D = -3w_1 - 2w_2$$

subject to

$$-w_1 - w_2 \geq 4$$
$$-w_1 + w_2 \geq 2$$
$$w_1, w_2 \geq 0.$$

Changing the objective function to maximization and introducing surplus variables: $w_3 \geq 0, w_4 \geq 0$ and artificial variables $w_{a_1}, w_{a_2} \geq 0$ the above dual problem reduces to

$$\text{Max. } Z_D' = 3w_1 + 2w_2 + 0w_3 + 0w_4 - Mw_{a_1} - Mw_{a_2}$$

subject to

$$-w_1 - w_2 - w_3 + w_{a_1} = 4$$
$$-w_1 + w_2 - w_4 + w_{a_2} = 2$$

Taking $w_1 = 0 = w_2 = w_3 = w_4$, we get $w_{a_1} = 4, w_{a_2} = 2$.

Now applying the simplex method to obtain the optimal solution, we have

	c_j	3	2	0	0			
B $\mathbf{c}_B$	$\mathbf{w}_B$	W_1	W_2	W_3	W_4	W_{a_1}	W_{a_2}	Min. ratio
W_{a_1} $-M$	4	-1	-1	-1	0	1	0	Negative
W_{a_2} $-M$	2	-1	[1]	0	-1	0	1	2 (min) →
$Z'_D = -6M$	Δ_j	$3-2M$	2 ↑	$-M$	- M	0	0 ↓	
W_{a_1} $-M$	6	-2	0	-1	-1	1	1	
W_2 **2**	2	-1	1	0	-1	0	1	
$Z'_D = 4-6M$	Δ_j	$5-2M$	0	$-M$	$2-M$	0	-2	

Since no $\Delta_j > 0$ and a non-zero artificial variable appears in the basis therefore the dual problem does not possess any optimum basic feasible solution.

Consequently, the given problem does not possess any finite optimal solution.

Example

Write the dual of the following linear programming problem and hence solve it.

$$\text{Max. } Z = 3x_1 - 2x_2$$

subject to

$$x_1 \leq 4$$
$$x_2 \leq 6$$
$$x_1 + x_2 \leq 5$$
$$-x_2 \leq -1$$
$$x_1, x_2 \geq 0.$$

The given problem is in standard primal form.

The dual of the given primal is

$$\text{Min. } Z_D = 4w_1 + 6w_2 + 5w_3 - w_4$$

subject to

$$w_1 + w_3 \geq 3$$
$$w_2 + w_3 - w_4 \geq -2$$
$$w_1, w_2, w_3, w_4 \geq 0$$

Changing the dual problem to maximization and introducing surplus variable w_5 and slack variable w_6 to change the inequalities into equations, the dual problem becomes:

$$\text{Max. } Z_D' = -4w_1 - 6w_2 - 5w_3 + w_4 + 0w_5 + 0w_6$$

subject to

$$w_1 + w_3 \qquad -w_5 = 3$$
$$-w_2 - w_3 + w_4 \quad +w_6 = 2$$

$$w_1, w_2, \ldots, w_6 \geq 0.$$

Taking $w_2 = 0, w_3 = 0$, we get $w_1 = 3, w_6 = 2$ which is the starting B.F.S.

The solution by simplex method is given in the following table :

	c_j	-4	-6	-5	1	0	0	Min. ratio
B c_B	w_B	W_1	W_2	W_3	W_4	W_5	W_6	w_B/W_4
W_1 -4	3	1	0	1	0	-1	0	Inf.
W_6 0	2	0	-1	-1	[1]	0	1	2 (min) →
$Z_D{}' = -12$	Δ_j	0	-6	-1	1 ↑	-4	0	
W_1 -4	3	1	0	1	0	-1	0	
W_4 1	2	0	-1	-1	1	0	1	
$Z_D{}' = -10$	Δ_j	0	-5	0	0	-4	-1	

The optimal solution of the dual is

$$w_1 = 3, w_2 = 0, w_3 = 0, w_4 = 2$$
$$\text{Min. } Z_D = -\text{ Max. } Z_D' = 10$$

The optimal solution of the primal problem is

$$x_1 = -\Delta_5 = 4, x_2 = -\Delta_6 = 1 \text{ and Max. } Z = \text{ Min. } Z_D = 10.$$