OPERATIONS RESEARCH

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What is Operation Research?

Scientific Approach

Scientific and mathematical approach to decision-making and problem-solving, employing rigorous methods.

Optimisation & Model

Uses models, statistics, and optimisation techniques to uncover the most effective solutions within intricate systems.

Broad Applications

Originating in WWII for military improvements, OR now finds widespread application across business, healthcare, logistics, and more.

Benefits of Operations Research

Cost Savings

Achieving significant cost reductions through precise optimisation of resource allocation and operational workflows.

Improved Productivity

Boosting efficiency by streamlining processes and refining existing workflows, ensuring maximum output with minimal input.

Risk Reduction

Minimising potential threats and uncertainties through robust forecasting models and proactive mitigation strategies.

Enhanced Customer Satisfaction

Elevating service quality and overall customer experience through thoughtful design and efficient delivery mechanisms.

Some Applications of OR

Supply Chain Optimisation

Reducing logistics costs and accelerating delivery times for goods.

Healthcare Scheduling

Improving patient flow, reducing wait times, and optimising resource utilisation in hospitals.

Transportation Planning

Optimising routes for public transport and logistics fleets, enhancing efficiency.

Financial Portfolio Management

Strategically balancing investment risk with potential returns to maximise financial gains.

The OR Problem-Solving Process

Define the Problem

Clearly articulate the issue to be addressed, including scope and boundaries.

Identify Objectives & Constraints

Establish clear goals and any limitations or restrictions that apply.

Develop a Model

Construct a mathematical or simulation model to represent the system.

Analyse Alternatives

Apply OR techniques to evaluate different solutions and their potential impacts.

Select Best Solution

Choose the most optimal course of action based on analysis and objectives.

Implement & Monitor

Put the solution into practice and continuously track its performance and results.

The Core Branches of Operations Research

Mathematical Optimisation

Identifying the absolute best solution within predefined constraints, as seen in **Linear Programming**.

Simulation

Modelling intricate systems to safely test various scenarios without exposing them to real-world risks.

Queuing Theory

Analysing waiting lines and customer flow patterns to significantly enhance service efficiency and throughput.

Decision Analysis & Game Theory

Thoroughly evaluating choices and strategic interactions to inform optimal decision-making in competitive environments.

Linear Programming Problem (LPP)

History

1945

War Ends

Military teams' success attracts industrial managers seeking solutions to complex executive problems.

1947

Breakthrough

George B. Dantzig develops the Simplex Method, the first mathematical technique for Linear Programming.

Post-1947

Evolution

New techniques and applications develop through collaboration between academic institutions and industry.

The Core Challenge

Central Question

What methods should be adopted so that total cost is minimum or total profit is maximum?

Mathematical Model

A simplified representation considering only the most important features for practical problem-solving.

Optimal Solution

Finding the most favourable solution that accounts for all circumstances and constraints.

Linear Programming

An optimization technique used in decision-making to obtain maximum or minimum values of a linear expression, subject to satisfying a specified number of linear restrictions.

Linear Programming Problem

The general problem calls for optimizing a linear function of variables (the objective function) subject to linear equations and/or inequations (the constraints).

Key Terminology

Linear

All relations governing the problem are linear—no products, powers, or complex combinations of variables are permitted.

Programming

The process of determining a particular programme or plan of action to achieve the desired objective.

Objective Function

The function to be optimized (maximized or minimized), representing profit, cost, time, or another measurable outcome.

Constraints

The system of linear inequations or equations under which the objective function must be optimized, reflecting real-world limitations.

Basic Requirements of an LPP

Well-Defined Objective

The objective function must be clear — either maximizing profit using available resources or minimizing cost with limited inputs.

Quantitative Measurement

Each element must be measurable quantitatively, with numerical data enabling relationships among elements to be established.

Constraints Present

Available resources to be allocated among competing activities must be limited, creating meaningful optimization challenges.

Alternative Actions

Multiple lines of action must exist for performing the task, allowing for genuine choice in decision-making.

Finite Constraints

The number of activities and constraints involved must be finite, ensuring the problem remains computationally tractable.

Mathematical Properties Required

Non-Negativity

All decision variables must assume non-negative values, reflecting real-world constraints where quantities cannot be negative.

Linearity

Both objective function and constraints must be linear.

(No products of variables ($x_1 x_2$), powers (x^2), or complex combinations permitted.)

Additivity

If machine M requires t_1 minutes for product A and t_2 minutes for product B, total time is $t_1 + t_2$. (assuming negligible changeover time)

Multiplicativity

If producing one unit costs £10, then producing x units costs £10x. Profit from selling n units equals profit per unit multiplied by n.

Divisibility

Fractional levels of variables must be permissible alongside integral values, allowing for continuous optimization.

Applications and Impact

Business & Commerce

Maximizing profit margins, optimizing product mix, and streamlining supply chains whilst respecting resource limitations.

Industry & Manufacturing

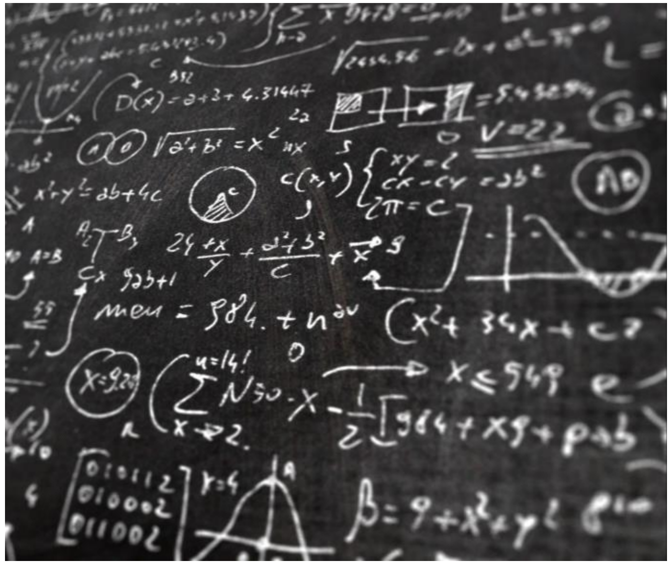
Minimizing production costs, scheduling operations efficiently, and allocating machinery and labour resources optimally.

Military Operations

Strategic resource allocation, tactical planning, and logistics optimization under constraints of limited assets.

Mathematical Formulation of LPP

Transforming real-world optimisation scenarios into rigorous mathematical models requires systematic identification of variables, objectives, and constraints. Mastering this translation from verbal description to mathematical formulation is essential for applying linear programming techniques effectively.



Standard Form

Find $x_1, x_2, \ldots, x_n$ to optimise

$$Z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n \qquad (1)$$

Subject to the linear constraints

$$\begin{array}{lcc}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & (\leq = \geq) & b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & (\leq = \geq) & b_2 \\
\ldots\ldots\ldots\ldots\ldots\ldots & \ldots & \ldots \\
\ldots & \ldots\ldots\ldots\ldots\ldots \quad \ldots & \ldots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & (\leq = \geq) & b_m
\end{array} \qquad (2)$$

and the non-negative restrictions

$$x_1, x_2, \ldots, x_n \geq 0,$$

where all $a_{11}, a_{12}, \ldots, a_{mn}; b_1, b_2, \ldots, b_m; c_1, c_2, \ldots, c_n$ are constants and $x_1, x_2, \ldots, x_n$ are variables.

Matrix Notation

Find $x_1, x_2, \dots, x_n$, to optimize

$$Z = \boldsymbol{c}\mathbf{x}$$

subject to

$$A\mathbf{x}\ (\leq = \geq)\ \mathbf{b}$$

and

$$\mathbf{x} \geq \mathbf{0}.$$

- $A = [a_{ij}]_{m \times n}$ is the **coefficient matrix**,
- $\boldsymbol{c} = [c_1, c_2, \dots, c_n]$ is the **price vector**,
- $\boldsymbol{b} = [b_1, b_2, \dots, b_n]'$ is the **requirement vector**,
- $\mathbf{x} = [x_1, x_2, \dots, x_n]'$ is the **decision variable vector**,
- $\mathbf{0}$ is Null matrix of size $n \times 1$.

Systematic Formulation Process

Step 1: Identify Decision Variables

Determine what quantities need to be decided and denote them systematically as $x_1, x_2, \dots, x_n$. These represent the controllable inputs to your problem.

Step 2: Define the Objective Function

Express the goal as a linear function of the variables. Formulate $z = c_1x_1 + c_2x_2 + \dots + c_nx_n$ where each coefficient c_j represents the contribution of variable x_j.

Step 3: Determine Optimisation Type

Identify whether the objective requires maximisation (e.g., profit, revenue, efficiency) or minimisation (e.g., cost, waste, time).

Step 4: Formulate Constraints

Identify all limitations and requirements, expressing each as a linear equation or inequality. Include resource limits, demand requirements, and capacity restrictions.

Example 1

A goldsmith manufactures necklaces and bracelets. The total number of necklaces and bracelets that he can handle per day is at most 24. It takes one hour to make a bracelet and half an hour to make a necklace. It is assumed that he can work for a maximum of 16 hours a day. Further the profit on a bracelet is Rs. 300 and the profit on a necklace is Rs. 100. Formulate this problem as a linear programming problem to maximize the profit.

Step 1: Profit Function

Goldsmith manufactures **Necklaces** and **Bracelets**

Let

$$x_1 = \text{ number of necklaces produced per day}$$

$$x_2 = \text{number of bracelets produced per day}$$

Profit per necklace = **Rs. 100**
Profit per bracelet = **Rs. 300**

Objective Function: $\text{Maximize } Z = 100x_1 + 300x_2$

Step 2: Time Constraint

Time to make one necklace = **0.5 hour**

Time to make one bracelet = **1 hour**

Total time required:

$$\frac{1}{2}x_1 + x_2 \ hours$$

Available time per day = **16 hours**

$$\frac{1}{2}x_1 + x_2 \leq 16$$

or equivalently,

$$x_1 + 2x_2 \leq 32$$

Step 3: Production Constraint

Maximum total items produced per day = **24**

$$x_1 + x_2 \leq 24$$

Step 4: Non-Negativity Conditions

$$x_1 \geq 0, x_2 \geq 0$$

Final LPP Formulation

$$\text{Maximize Z} = 100x_1 + 300x_2$$

subject to

$$\begin{aligned} x_1 + 2x_2 &\leq 32 \\ x_1 + x_2 &\leq 24 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Example 2

A tyre factory produces three types of tyres T1, T2, T3. Three different types of chemicals say C1, C2, C3 are required for production. One T1 tyre needs 2 units of C1, 3 units of C3; one T2 tyre needs 3 units of C1, 2 units of C2 and 2 units of C3; and one T3 tyre needs 5 units of C2 and 4 units of C3. The factory has only a stock of 20 units of C1, 25 units of C2 and 30 units of C3. Further the profit from the sale of one tyre T1 is Rs. 6, one tyre T2 is Rs. 10 and of one tyre T3 is Rs. 8. Assuming the factory can sell all the tyres it produces, formulate a linear programming problem to maximize its profits.

A tyre factory produces three types of tyres:

T1, T2, T3

Three chemicals required:

C1, C2, C3

Let

x_1 =number of tyres T_1

x_2 =number of tyres T_2

x_3 =number of tyres T_3

Chemical Requirement Table

Tyres	C1	C2	C3
T1	2	0	3
T2	3	2	2
T3	0	5	4

Available stock

$C1 = 20$ units

$C2 = 25$ units

$C3 = 30$ units

Chemical Constraints:

For **C1 :**

$$2x_1 + 3x_2 \leq 20$$

For **C2 :**

$$2x_2 + 5x_3 \leq 25$$

For **C3 :**

$$3x_1 + 2x_2 + 4x_3 \leq 30$$

Profit per tyre T_1 =Rs. 6

Profit per tyre T_2 =Rs. 10

Profit per tyre T_3 =Rs. 8

Objective Function:

$$\text{Maximize } Z = 6x_1 + 10x_2 + 8x_3$$

Non-Negativity Conditions

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

Final LPP Formulation

$$Maximize\ \mathrm{Z} = 6x_1 + 10x_2 + 8x_3$$

subject to

$$\begin{aligned} 2x_1 + 3x_2 &\leq 20 \\ 2x_2 + 5x_3 &\leq 25 \\ 3x_1 + 2x_2 + 4x_3 &\leq 30 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Example 3

The objective of a diet problem is to ascertain the quantities of certain foods that should be eaten to meet certain nutritional requirement at a minimum cost. The consideration is limited to milk, green vegetables and eggs, and to vitamins A, B, C. The number of milligrams of each of these vitamins contained within a unit of each food and their daily minimum requirement along with the cost of each food is given in the table below :

Vitamin	Litre of Milk	Kg. of Vegetables	Dozen of Eggs	Minimum Daily Requirement
A	1	1	10	1 mg
B	100	10	10	50 mg
C	10	100	10	10 mg
Cost in Rs.	20	10	8	

Formulate a linear programming problem for this diet problem.

Let daily diet consists of

x_1 litres of **milk**

x_2 kilograms of **vegetables**

x_3 dozens of **eggs**

The objective is to **minimize the daily cost** while meeting **vitamin requirements**.

Cost Information:

Cost per unit

Milk: Rs. 20 per litre

Vegetables: Rs. 10 per kg

Eggs: Rs. 8 per dozen

Objective Function:

$$Z = 20x_1 + 10x_2 + 8x_3$$

Vitamin A Requirement

Total Vitamin A in the diet:

$$x_1 + x_2 + 10x_3 \; mg$$

Minimum required Vitamin A = **1 mg**

$$x_1 + x_2 + 10x_3 \geq 1$$

Vitamin B Requirement

Total Vitamin B in the diet:

$$100x_1 + 10x_2 + 10x_3 \; mg$$

Minimum required Vitamin B = **50 mg**

$$100x_1 + 10x_2 + 10x_3 \geq 50$$

Vitamin C Requirement

Total Vitamin C in the diet:

$$10x_1 + 100x_2 + 10x_3 \; mg$$

Minimum required Vitamin C = **10 mg**

$$10x_1 + 100x_2 + 10x_3 \geq 10$$

Non-Negativity Conditions

The quantities of food items cannot be negative.

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

Final LPP Formulation:

$$minimize\ Z = 20x_1 + 10x_2 + 8x_3$$

subject to

$$\begin{aligned} x_1 + x_2 + 10x_3 &\geq 1, \\ 100x_1 + 10x_2 + 10x_3 &\geq 50, \\ 10x_1 + 100x_2 + 10x_3 &\geq 10, \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

Example 4

A manufacturer produces three models (I, II and III) of a certain product. He uses two types of raw materials (A and B) of which 4000 and 6000 units respectively are available. The raw material requirements per unit of the three models are given below:

Raw Material	Requirement per unit of given model		
	I	II	III
A	2	3	5
B	4	2	7

The labour time for each unit of model I is twice that of model II and three times that of model III. The entire labour force of the factory can produce the equivalent of 2500 units of model I. A market survey indicates that the minimum demand of the three models are 500, 500 and 375 units respectively. However, the ratio of the number of units produced must be equal to 3: 2: 5. Assume that the profits per unit of models I, II and III are rupees 60,40 and 100 respectively. Formulate the problem as an LPP in order to determine the number of units of each product which will maximize profit.

Let the manufacturer produce

x_1 units of **Model I**

x_2 units of **Model II**

x_3 units of **Model III**

Profit per unit:

Model I → Rs. 60

Model II → Rs. 40

Model III → Rs. 100

Objective: Maximize total profit

$$\boxed{Z = 60x_1 + 40x_2 + 100x_3}$$

Raw Material Constraints

Based on availability of raw materials:

$$2x_1 + 3x_2 + 5x_3 \leq 4000$$
$$4x_1 + 2x_2 + 7x_3 \leq 6000$$

Market Demand Constraints

Minimum demand requirements:

$$x_1 \geq 500$$
$$x_2 \geq 500$$
$$x_3 \geq 375$$

Labour Time Constraint:

Let labour time for **Model I** = t. Then time for

$$\text{Model II} = \frac{t}{2}$$

$$\text{Model III} = \frac{t}{3}$$

Maximum production capacity = **2500 units of Model I**

$$tx_1 + \frac{t}{2}x_2 + \frac{t}{3}x_3 \leq 2500t$$

Dividing by t and simplifying

$$\boxed{6x_1 + 3x_2 + 2x_3 \leq 15000}$$

Production Ratio Constraint:

Given production ratio:

$$x_1 : x_2 : x_3 = 3 : 2 : 5$$

Let

$$x_1 = 3k, \quad x_2 = 2k, \quad x_3 = 5k$$

Equivalent constraint equations:

$$\frac{1}{3}x_1 = \frac{1}{2}x_2$$

$$\frac{1}{2}x_2 = \frac{1}{5}x_3$$

Final LPP Model

$$\text{Maximize } Z = 60x_1 + 40x_2 + 100x_3$$

subject to

$$2x_1 + 3x_2 + 5x_3 \leq 4000$$

$$4x_1 + 2x_2 + 7x_3 \leq 6000$$

$$6x_1 + 3x_2 + 2x_3 \leq 15000$$

$$2x_1 = 3x_2$$

$$5x_2 = 2x_3$$

$$x_1 \geq 500,\ \ x_2 \geq 500,\ \ x_3 \geq 375$$

Exercise

A city hospital has the minimal daily requirements for nurses as given in the Table. Nurses report to the hospital at the beginning of each period and work for 8 consecutive hours. The hospital wants to determine the minimum number of nurses available for each period.

Formulate this as an LPP by setting up appropriate constraints and objective function.

Period	Clock time (24 hr. day)	Minimum number of nurses required
1	6 A.M. - 10 A.M.	2
2	10 A.M. - 2 P.M.	7
3	2 P.M. - 6 P.M.	15
4	6 P.M. - 10 P.M.	8
5	10 P.M. - 2 A.M.	20
6	2 A.M. - 6 A.M.	6

Solution of LPP

What is a Solution?

Solution of an LPP refers to a set of values for the decision variables that satisfies all the given constraints of the problem.

These constraints typically represent limitations on resources, production capacities, or other operational boundaries.

Visualising Constraints

Each constraint in an LPP defines a region in the solution space. A solution is simply a point that falls within the allowable zone dictated by these inequalities.

The Concept of a Feasible Solution

A 'feasible solution' is a more refined concept. It includes not only satisfying all the problem's constraints but also adhering to 'non-negative restrictions'. These typically mean that decision variables (e.g., quantities of products) cannot be negative.

Practical Implications

In real-world applications, feasible solutions represent realistic and achievable plans. For instance, you cannot produce a negative number of items or allocate negative resources.

Optimal Solution

Finding the Best Outcome

An 'optimal solution' is a feasible solution that either maximises (e.g., profit) or minimises (e.g., cost) the objective function of the LPP.

This is the ultimate aim of solving any linear programming problem.

It represents the most efficient or effective outcome given the constraints and objectives.

Unbounded Solutions

Infinite Improvement

An 'unbounded solution' occurs when the value of the objective function can be increased (for maximisation problems) or decreased (for minimisation problems) indefinitely without violating any constraints.

Indication of Problem Formulation Issues

This usually suggests that the problem has been incorrectly formulated, as real-world scenarios always have some limits or boundaries. It highlights the importance of thorough constraint definition.

Different Approaches to Solve LPP

Graphical Method

Ideal for problems with two variables, offering a visual solution.

Analytic Method

A trial-and-error approach for problems exceeding two variables.

Simplex Method

The most versatile and powerful algebraic procedure.

Steps to Graph Simultaneous Inequations

1

2

3

Graph Each Inequation

For each linear inequation, first graph the corresponding equality (a straight line). Then, determine which side of the line represents the solution for that specific inequation.

Identify Overlapping Region

The 'solution set' for the system is the region where all individual shaded areas (solutions for each inequation) overlap. This common region is often referred to as the feasible region.

Consider Boundaries

The lines forming the boundaries of this common region are crucial, especially their intersection points, which are the extreme points mentioned earlier.

Possible Outcomes of the Solution Set

Empty Set

No common region satisfies all inequations. This indicates no feasible solution exists.

Bounded Region

The solution set is a closed, finite area, typically a polygon. This is common in well-defined LPPs.

Unbounded Region

The solution set extends infinitely in one or more directions. This can lead to unbounded objective function values.

The Graphical Method

- provides an intuitive way to solve LPPs when the objective function depends on only two variables.

- simplifies complex relationships into a visual model.

- provides clear insight into the feasible region.

While it can theoretically handle three variables, the complexity quickly escalates, making it less practical for higher dimensions.