OPERATIONS RESEARCH

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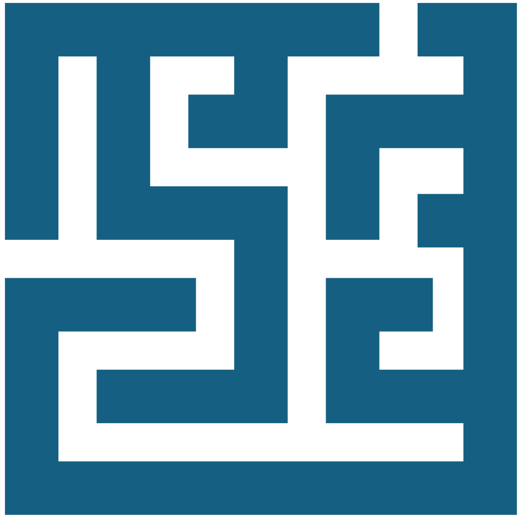
The Corner-Point Method



The Corner-Point Method is based on the extreme point theorem. It involves identifying the "feasible region”.

- **Feasible Region:** The set of all possible solutions that adhere to the problem's constraints.

- **Convex Polygon:** The shape formed by this feasible region, also known as the permissible region.

The vertices of this polygon are critical, as they represent the extreme points where optimal solutions often reside.

The Fundamental Extreme Point Theorem

This theorem significantly simplifies the search for an optimal solution, narrowing it down to a finite number of points on the boundary of the feasible region.

Statement:

An optimum solution of an LPP, if it exists, occurs at one of the extreme points (i.e., corner points) of the convex polygon formed by the set of all feasible solutions.

Identifying Optimal Solutions

- Once the feasible region is established, the next step is to evaluate the objective function at each of its corner points.
- The point yielding the maximum or minimum value (depending on whether optimising for profit or cost) is the optimal solution.

Note: If two vertices yield the same optimal value, all points along the line segment connecting them represent optimal solutions, indicating infinite possibilities.

Procedure for Corner-Point Method

Step 1: Convert Constraints

Treat each inequality constraint as a linear equation.

Step 2: Plot the Lines

Draw these equations on the coordinate plane. Remember to also account for non-negative restrictions.

Step 3: Identify Feasible Region

Determine the area where all constraints are simultaneously satisfied. This is your permissible region, a convex polygon.

Step 4: Pinpoint Vertices

Determine the coordinates of all the corner points (vertices) of the feasible region. These are your extreme points.

Step 5: Evaluate Objective Function

Substitute the coordinates of each vertex into the objective function. The point yielding the desired optimum (maximum or minimum) is your solution.

Limitations of the Graphical Method

Two-Variable Restriction

Exclusively applicable to problems with only two decision variables.

Real-World Applicability

Most real-world LPPs involve numerous variables, rendering the graphical method unsuitable for direct solution.

Increased Complexity

Even with three variables, visualisation becomes highly complex and often impractical.

Computational Demands

For large-scale problems, manual graphing is time-consuming and prone to error.

Example 1

Solve by graphical method, the linear programming problem

$$\text{Minimize } Z = 20x_1 + 10x_2$$

subject to the constraints,

$$\begin{gathered} x_1 + 2x_2 \leq 40 \\ 3x_1 + x_2 \geq 30 \\ 4x_1 + 3x_2 \geq 60 \end{gathered}$$

and the non-negative restrictions $x_1, x_2 \geq 0$.

Solution

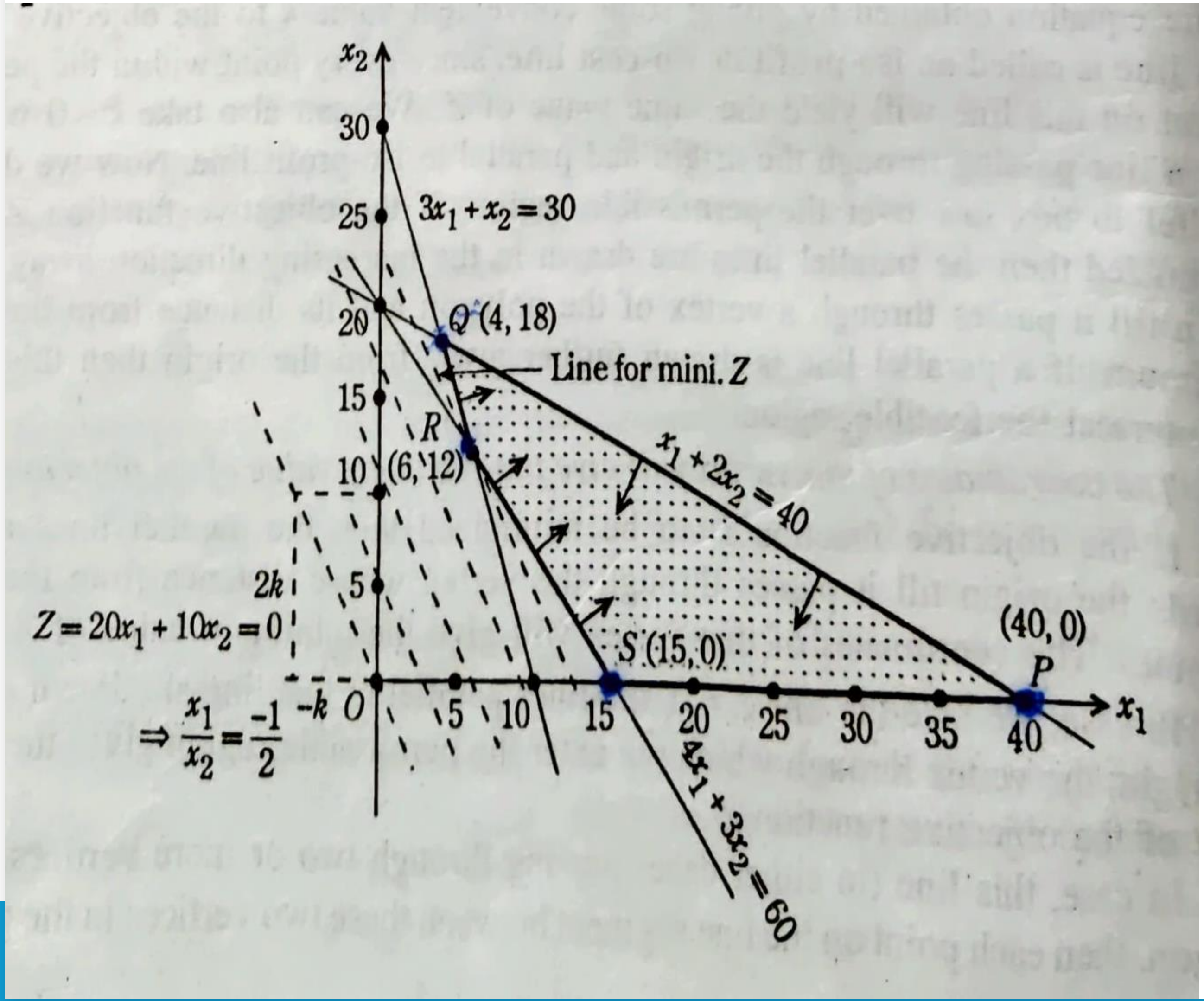
Step 1: Convert Constraints into Equations

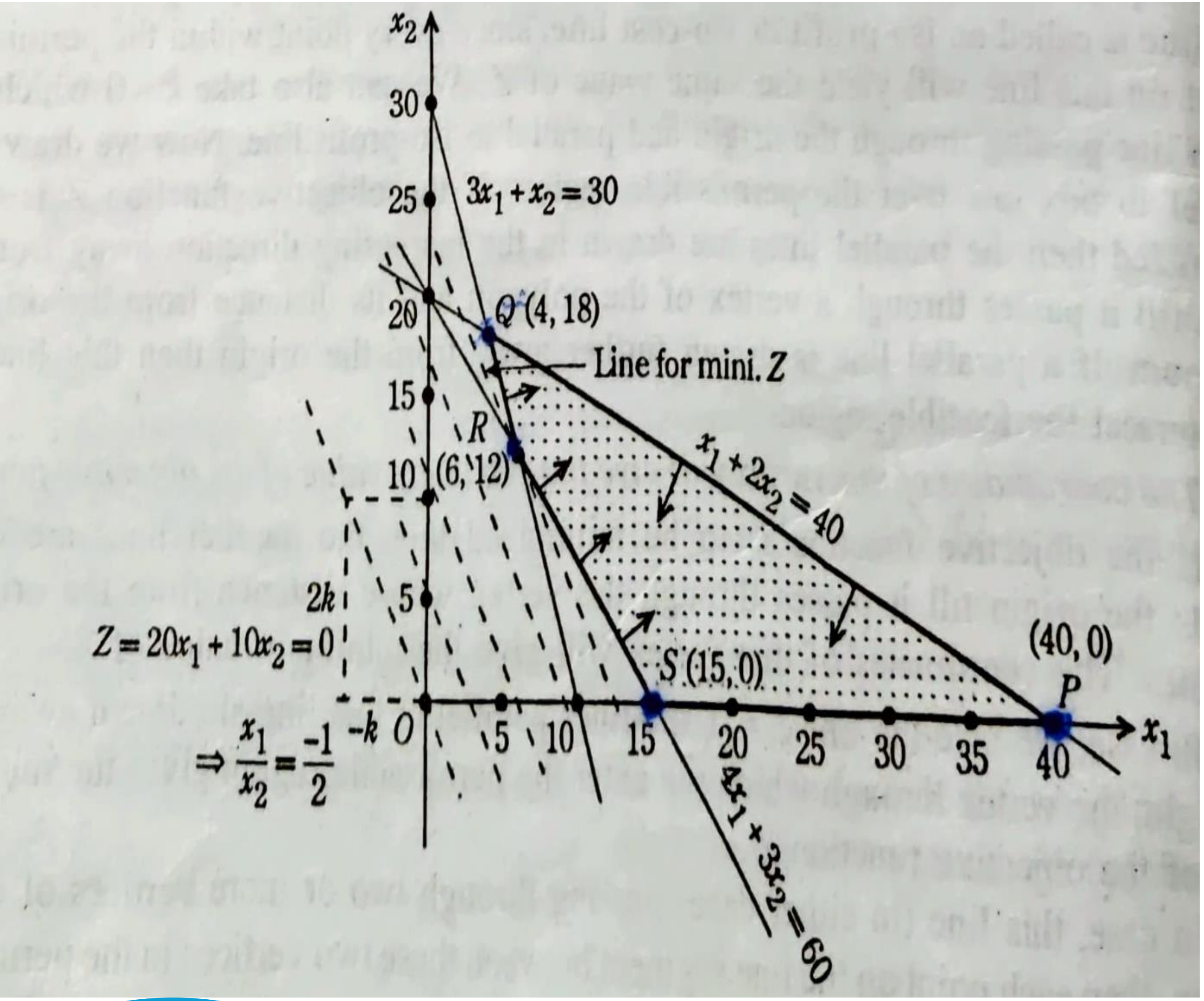
The given constraints are written as:

- $x_1 + 2x_2 = 40$
- $3x_1 + x_2 = 30$
- $4x_1 + 3x_2 = 60$

Step 2: Plot the Constraint Lines

- Draw straight lines corresponding to each equation on the $x_1 x_2$-plane.
- Identify the common shaded region satisfying all constraints.





Step 3: Feasible (Permissible) Region

- The shaded region **PQRSP** represents the feasible region.
- It is a convex polygon.

Step 4: Corner-Point Method

Find the coordinates of the vertices by solving intersecting lines

- $P(40,0)$
- $Q(4,18)$
- $R(6,12)$
- $S(15,0)$

- **Step 5: Evaluate Objective Function**

$$Z = 20x_1 + 10x_2$$

Point (x, y)	Value of the objective function $Z = 20x_1 + 10x_2$
$P(40,0)$	$Z = 20 \times 40 + 10 \times 0 = 800$
$Q(4,18)$	$Z = 20 \times 4 + 10 \times 18 = 260$
$R(6,12)$	$Z = 20 \times 6 + 10 \times 12 = 240$ (Min.)
$S(15,0)$	$Z = 20 \times 15 + 0 = 300$

Optimal Solution (Corner-Point Method)

$$x_1 = 6, \quad x_2 = 12$$

Minimum value : $Z = 240$

Example 2

A dietician mixes two types of food in such a way that the vitamin contents of the mixture contain at least 8 units of vitamin A and 10 units of vitamin C. Food X contains 2 units/kg of vitamin A and 1 unit/kg of vitamin C while food Y contains 1 unit/kg of vitamin A and 2 units/kg of vitamin C. One kg of food X costs Rs. 5 whereas one kg of food Y costs Rs. 7. Determine the minimum cost of such a mixture.

The mathematical form of the given problem as an LPP is as follows:

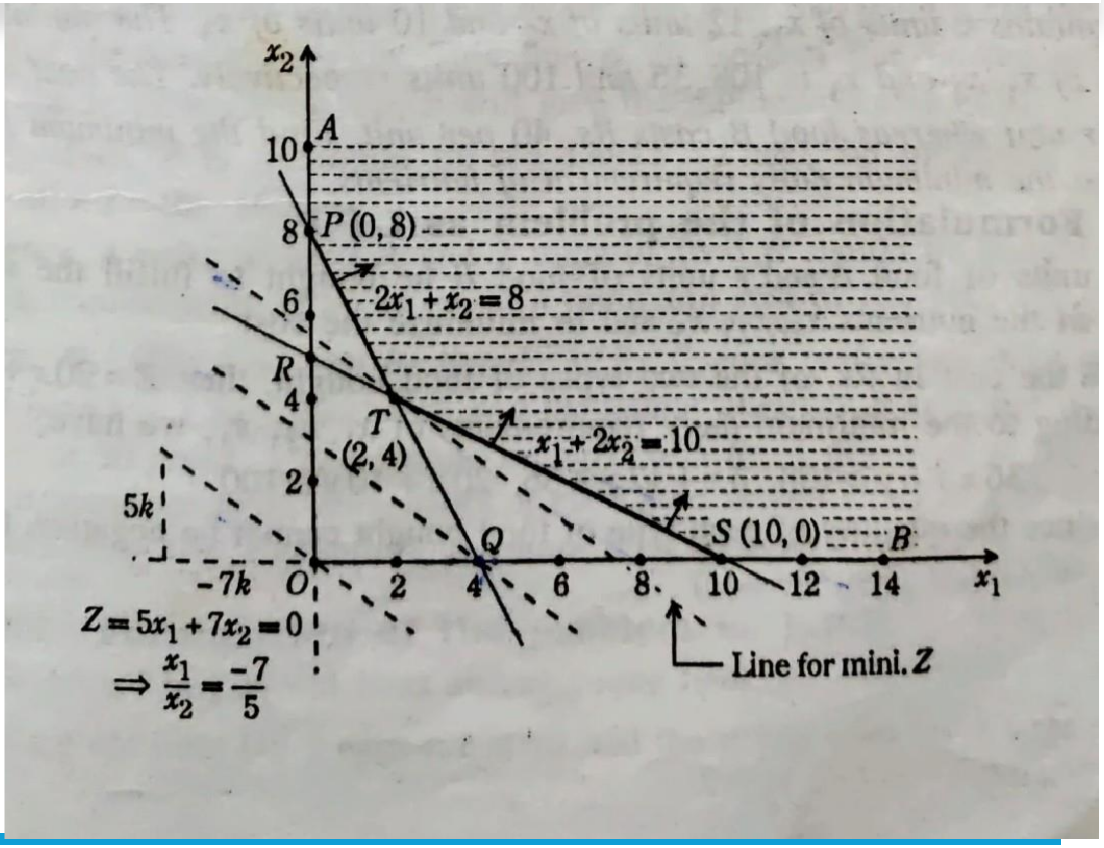
$$\text{Minimize } Z = 5x_1 + 7x_2$$

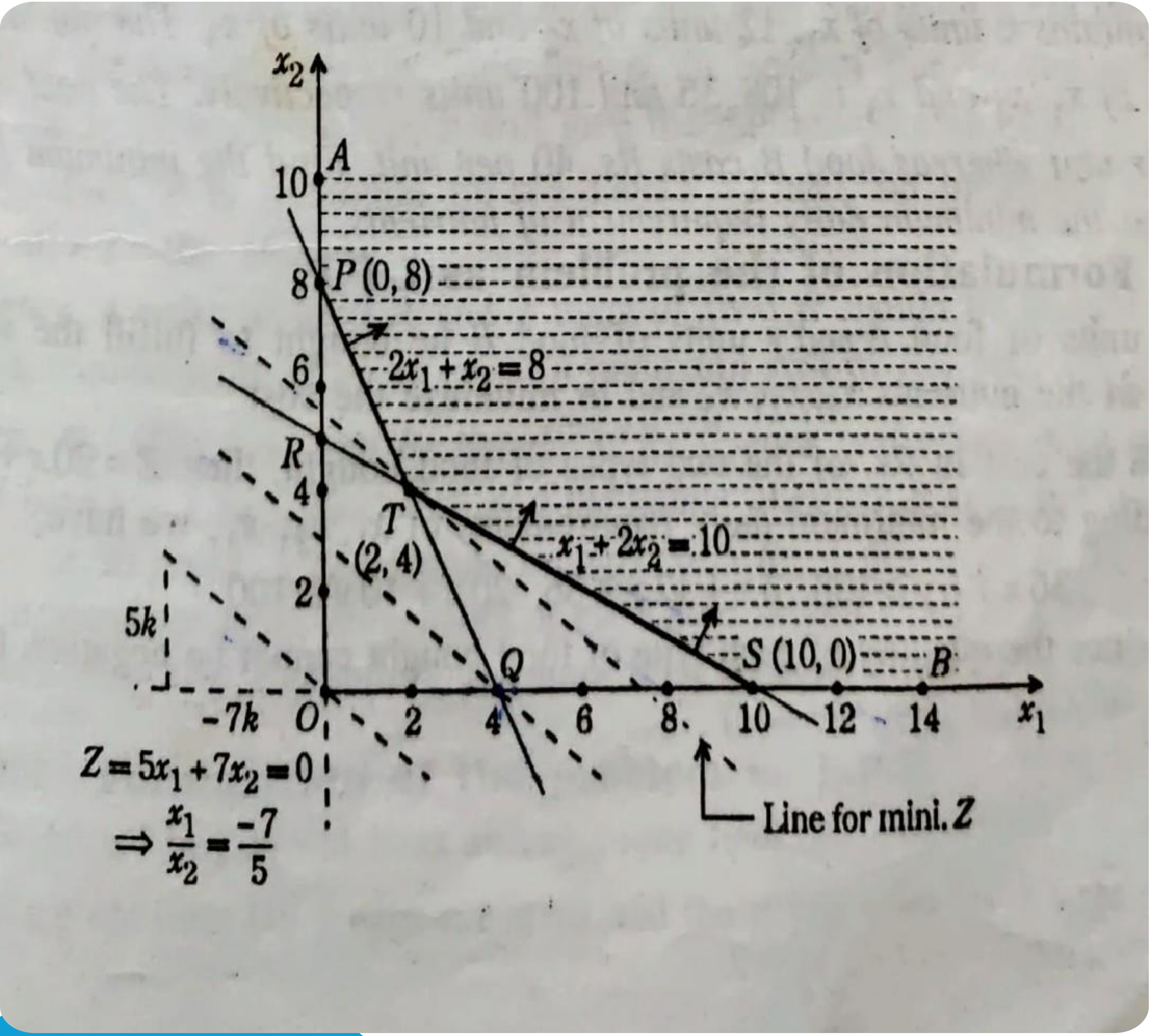
subject to the constraints

$$2x_1 + x_2 \geq 8$$

$$x_1 + 2x_2 \geq 10$$

and the non-negative restrictions.





Feasible (Permissible) Region

- The **permissible region** is the **shaded region**.
- It represents the **set of all feasible solutions** of the given L.P.P.
- The permissible region is **unbounded**.

Corner Points of the Permissible Region

$$P(0,8), \qquad T(2,4), \qquad S(10,0)$$

Evaluation of Objective Function

Point (x_1, x_2)	Value of the objective function $Z = 5x_1 + 7x_2$
$P(0,8)$	$Z = 5 \times 0 + 7 \times 8 = 56$
$T(2,4)$	$Z = 5 \times 2 + 7 \times 4 = 38$ (Min.)
$S(10,0)$	$Z = 5 \times 10 + 7 \times 0 = 50$

Optimal Solution

The minimum value of the objective function occurs at point **T(2,4)**.

$$x_1 = 2, x_2 = 4$$

Conclusion

- The dietician should mix:
  - **2 kg of food X**
  - **4 kg of food Y**
- Minimum cost:

$$\boxed{Z = 38\ rupees}$$

Example 3

Suresh has two factories of toys: one located in city X and the other in city Y. Both these factories manufacture the same type of toys. From these locations a certain number of toys are delivered to each of the three depots situated at places A, B, and C. The weekly requirements of the depots are respectively 5,5 and 4 units, while the production capacity of the factories at X and Y are respectively 8 and 6 units. The transportation cost in Rs. per unit from a factory to a depot is as given in the table.

	A	B	C
X	16	10	15
Y	10	12	10

How many units should be transported from each factory to each depot in order that the transportation cost is minimum ?

Mathematical Formulation of Problem

Decision Variables

Let

x_1 units of toys transported from factory **X** to depot **A**

x_2 units of toys transported from factory **X** to depot **B**

Supply from Factory X

- Production capacity of factory **X** = 8 units
- Units supplied to depot **C** from factory X:

$$8 - x_1 - x_2$$

Non-negativity Constraints

Since the number of units transported cannot be negative:

$$x_1 \geq 0, x_2 \geq 0$$

$$8 - x_1 - x_2 \geq 0 \Rightarrow x_1 + x_2 \leq 8$$

Demand at Depot A

- Weekly requirement of depot **A** = 5 units
- Units supplied from factory **Y** to depot A:

$$5 - x_1$$

Non-negativity gives:

$$5 - x_1 \geq 0 \Rightarrow x_1 \leq 5$$

Demand at Depot B

- Weekly requirement of depot **B** = 5 units
- Units supplied from factory **Y** to depot B:

$$5 - x_2$$

Non-negativity gives:

$$5 - x_2 \geq 0 \Rightarrow x_2 \leq 5$$

Supply from Factory Y

Production capacity of factory **Y** = 6 units
Units supplied from factory Y to depot **C**:

$$6 - [(5 - x_1) + (5 - x_2)] = x_1 + x_2 - 4$$

Non-negativity condition:

$$x_1 + x_2 - 4 \geq 0 \Rightarrow x_1 + x_2 \geq 4$$

Objective Function (Transportation Cost)

Total transportation cost (in Rs.)

$$Z = 16x_1 + 10x_2 + 15(8 - x_1 - x_2)$$
$$+10(5 - x_1) + 12(5 - x_2) + 10(x_1 + x_2 - 4)$$

Simplifying:

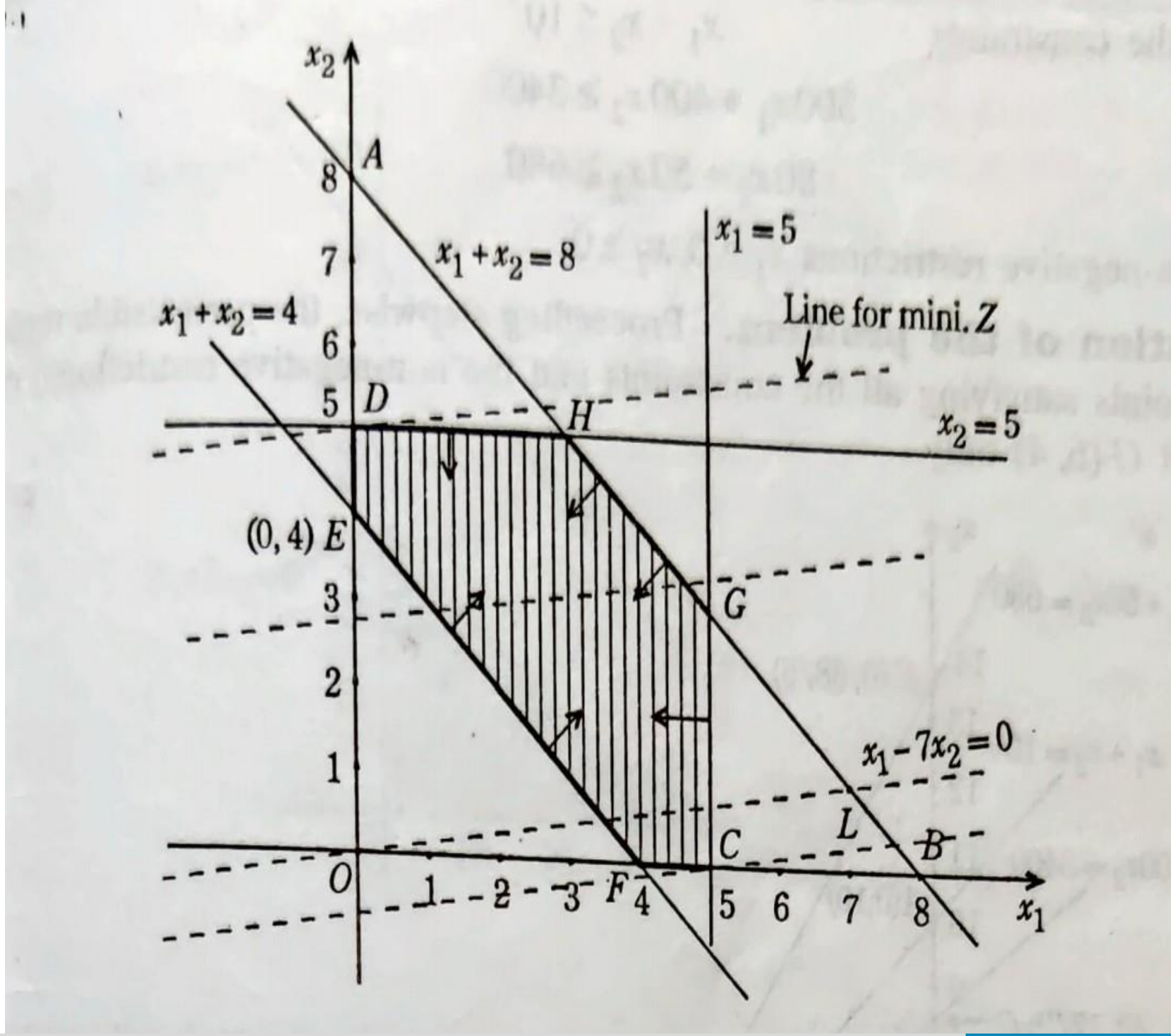
$$Z = x_1 - 7x_2 + 190$$

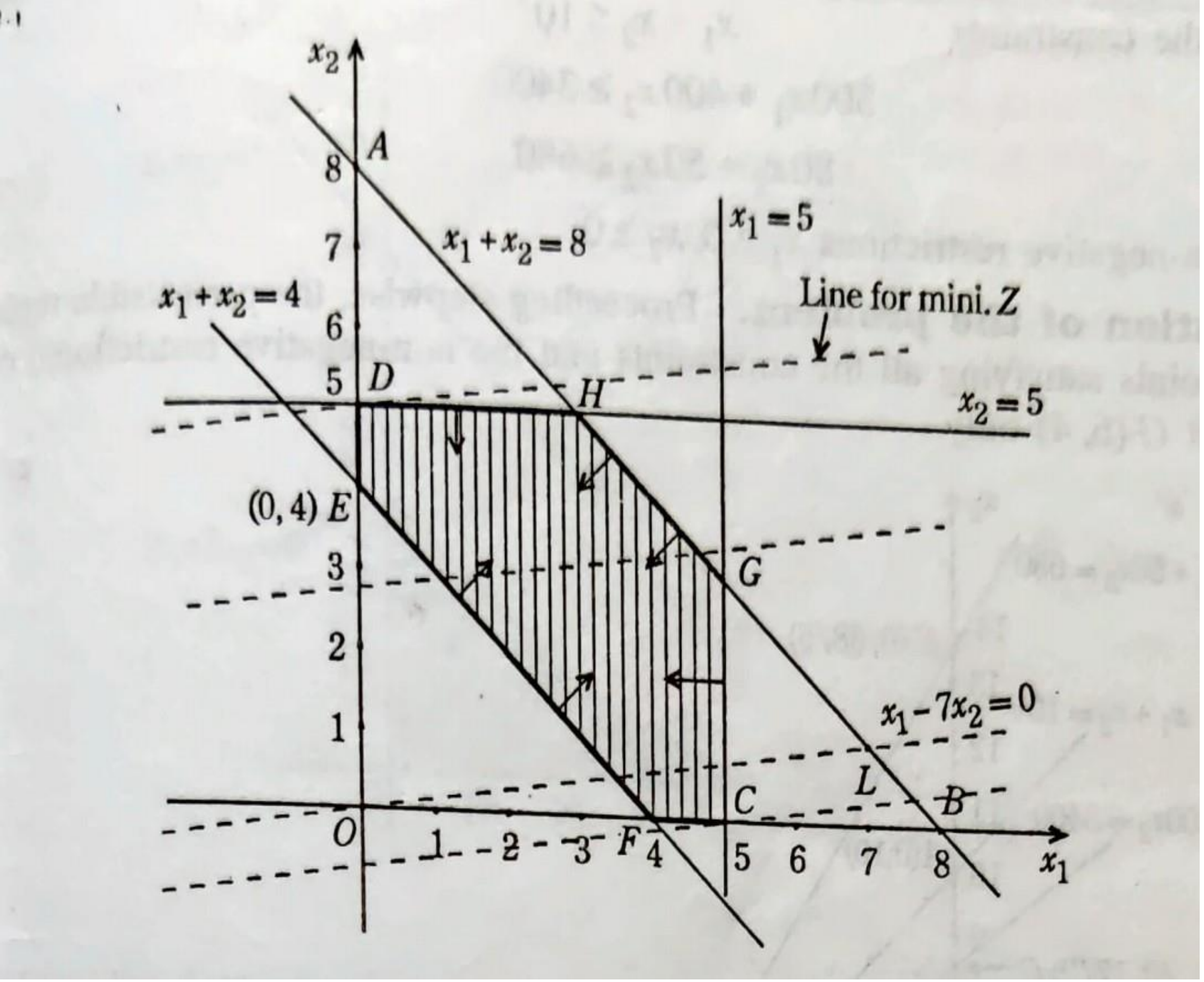
Formulation of the LPP

$$\textbf{\textit{Minimize}}\ \ Z = x_1 - 7x_2 + 190$$

Subject to constraints

$$\begin{aligned} x_1 + x_2 &\le 8 \\ x_1 &\le 5 \\ x_2 &\le 5 \\ x_1 + x_2 &\ge 4 \\ x_1 \ge 0,\ x_2 &\ge 0 \end{aligned}$$





Feasible (Permissible) Region

- The permissible region is the **shaded region EFCGHDE**.
- The feasible region is **bounded**.

Corner Points of the Feasible Region

$$E(4,0),\ F(5,0),\ C(5,3),\ G(3,5),\ H(0,5),\ D(0,4)$$

Evaluation of Objective Function at Corner Points

Corner Point	Points (x_1, x_2)	Value of $Z = x_1 - 7x_2 + 190$
E	(4, 0)	$Z = 4 - 0 + 190 = 194$
F	(5, 0)	$Z=5 - 0 + 190 = 195$
C	(5, 3)	$Z=5 - 21 + 190 = 174$
G	(3, 5)	$Z=3 - 35 + 190 = 158$
H	(0, 5)	$Z=0 - 35 + 190 = 155$ **(Minimum)**
D	(0,4)	Z=0-28+190=162

Optimal Solution (Corner-Point Method)

Minimum value of the objective function occurs at $H(0,5)$

$$x_1 = 0, \quad x_2 = 5, \quad Z_{\min} = 155$$

Optimal Transportation Strategy

- **From factory X** :

To **A**: 0 units, To **B**: 5 units, To **C**: 3 units

- **From factory Y** :

To **A**: 5 units, To **B**: 0 units, To **C**: 1 unit

Example 4

Solve by graphical method the L.P.P.

$$\text{Minimize } Z = 5x_1 + 6x_2$$

subject to the constrains

$$x_1 + x_2 \geq 50, x_1 + 2x_2 \leq 40,$$

$$3x_1 + 4x_2 \leq 100$$

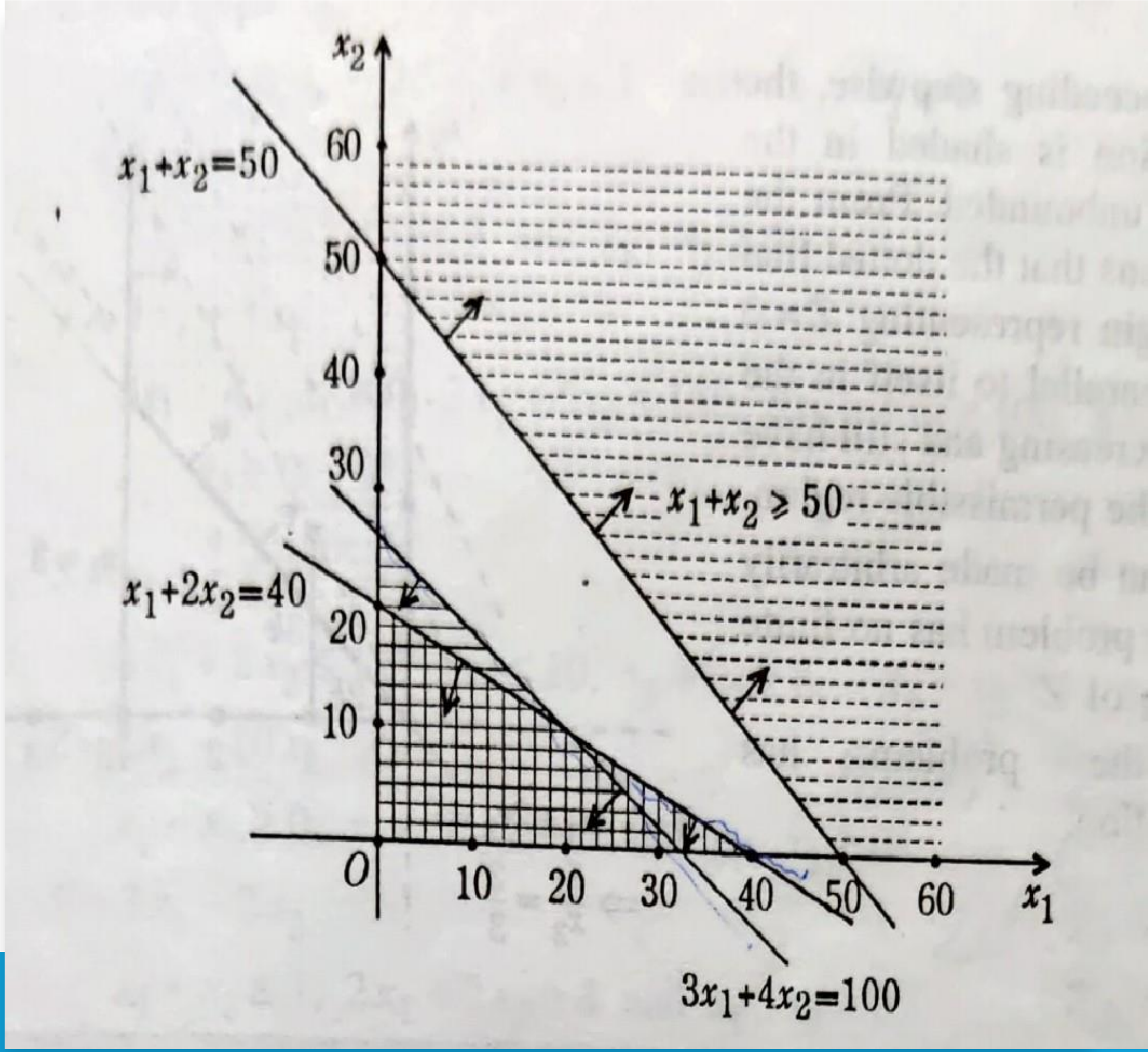
and the non-negative restrictions

$$x_1 \geq 0 \text{ and } x_2 \geq 0.$$

- There exist **no values** of x_1 and x_2 which **simultaneously satisfy** all the **constraints** along with the **non-negative restrictions**.

- The feasible (permissible) region is empty, i.e., **no feasible solution exists.**

- Such a problem is called an **infeasible linear programming problem**.



Example 5

Solve graphically the following L.P.P.

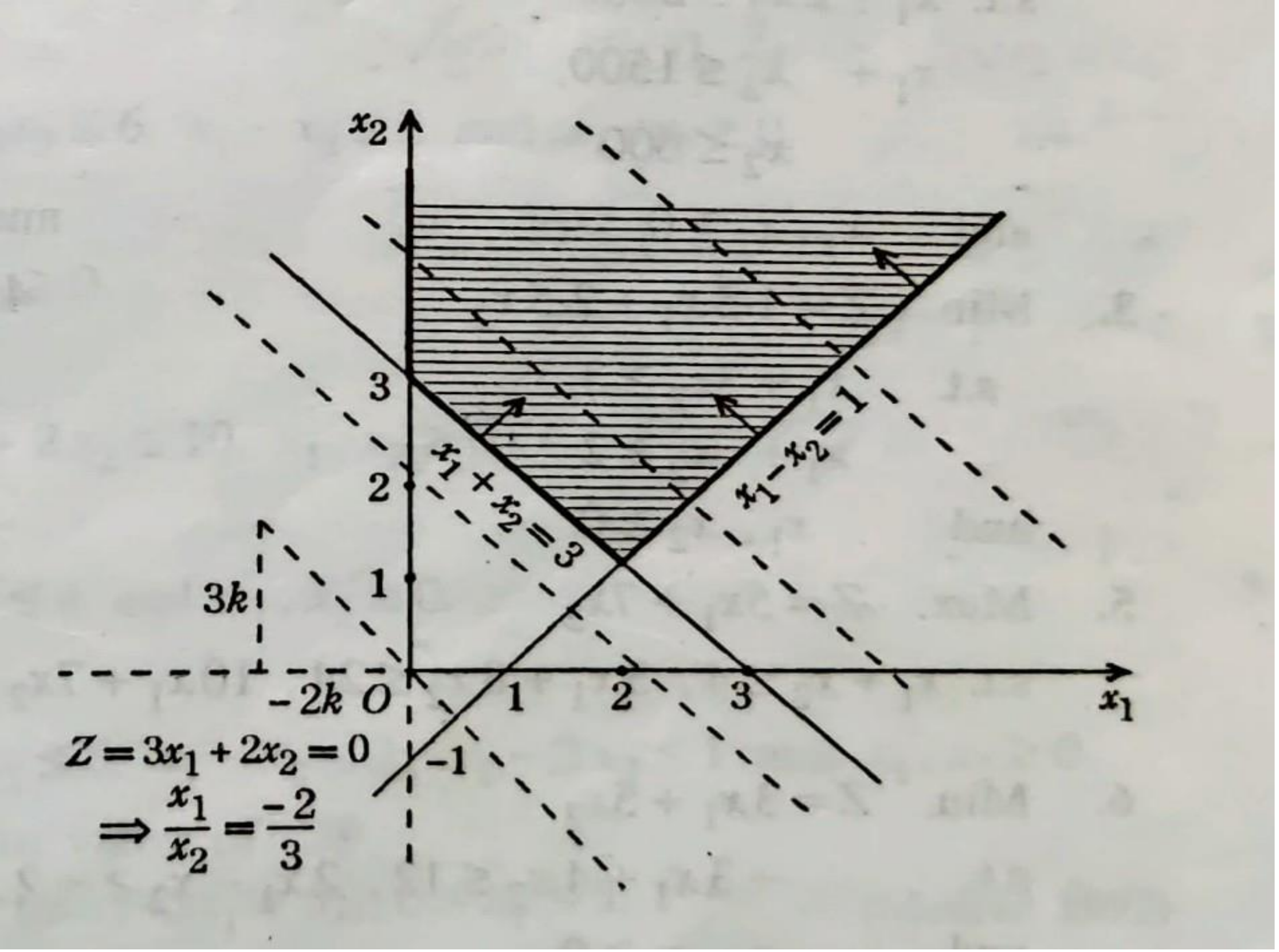
$$Max.\,Z = 3x_1 + 2x_2$$

s.t.

$$x_1 - x_2 \leq 1$$
$$x_1 + x_2 \geq 3$$

and

$$x_1, x_2 \geq 0$$



- The **objective function line** (dotted line through origin, Z=0) can move **parallel in the direction of increasing Z**.

- Movement still **intersects the feasible region**.

Conclusion:

- Z can be made **arbitrarily large**.
- Therefore, **no finite maximum value** exists.
- The linear programming problem is **unbounded**.

Exercise

Solve graphically the following linear programming problems:

$$Max.Z = x_1 + x_2$$

s.t

$$x_1 + 2x_2 \leq 2000,$$

$$\begin{aligned} x_1 + x_2 &\leq 1500 \\ x_2 &\leq 600 \end{aligned}$$

and

$$x_1, x_2 \geq 0$$

$$Max.Z = 0.75x_1 + x_2$$

s.t.

$$x_1 - x_2 \geq 0,$$

$$-0.5x_1 + x_2 \leq 1$$

and

$$x_1, x_2 \geq 0.$$

Basic Solutions

Consider a system of linear equations

$$A\mathbf{x} = \mathbf{b},$$

with m equations and n unknowns where $n > m$ and rank conditions

$$r(A) = r(A\mathbf{b}) = m$$

hold, ensuring no redundant equations .

- Choose any $(n - m)$ variables and set them to zero.
- Ensure the determinant of coefficients of remaining m variables is non-zero.

A solution obtained using this setting is called **Basic Solution.**

Basic variables: The m variables whose coefficients form a non-singular matrix. They can be zero or non-zero.

Non-Basic Variables: The $(n - m)$ variables that are explicitly set to zero to obtain the basic solution.

Key Properties of Basic Solutions

Linear Independence

- The matrix B formed by coefficients of basic variables is non-singular. Therefore, vectors associated with basic variables are linearly independent.
- The basic solution is computed as

$$B\mathbf{x}_B = \mathbf{b} \quad \text{or} \quad \mathbf{x}_B = B^{-1}\mathbf{b}$$

Counting Solutions

- The maximum number of basic solutions is determined by selecting m linearly independent vectors from n available vectors.
- This yields at most $^{\boldsymbol{n}}\boldsymbol{C}_{\boldsymbol{m}} = \frac{\boldsymbol{n}!}{\boldsymbol{m}!(\boldsymbol{n}-\boldsymbol{m})!}$ possible basic solutions.

Types of Basic Solutions

Non-Degenerate Basic Solution

- A basic solution where **none of the basic variables equals zero**.
- Contains exactly m non-zero variables and $(n - m)$ zero variables.
- Provides unique and well-defined solution points in the feasible region.

Degenerate Basic Solution

- A basic solution where **at least one basic variable equals zero**.
- Creates computational challenges in optimization algorithms.
- May correspond to multiple basic solutions at the same point.

Existence and Non-Degeneracy Theorem

Theorem

A necessary and sufficient condition for the existence and non-degeneracy of all basic solutions of

$$A\mathbf{x} = \boldsymbol{b}$$

is that every set of m columns of the augmented matrix $[A, \boldsymbol{b}]$ is linearly independent.

Necessity

If all basic solutions exist and are non-degenerate, then **b** can replace any column in a basis, maintaining linear independence

Sufficiency

If every m-column set of $[A,\mathbf{b}]$ is linearly independent, then all basic solutions exist and are non-degenerate

Basic Feasible Solutions in Linear Programming

A **basic feasible solution (BFS)** is a feasible solution that is also basic—meaning vectors associated with non-zero variables are linearly independent.

- **At Most $\boldsymbol{m}$ Non-Zero Variables** Since only m vectors can be linearly independent in a system with m constraints

- **At Least $(\boldsymbol{n}-\boldsymbol{m})$ Zero Variables** For a feasible solution to qualify as a BFS, at least $(n-m)$ variables must vanish

- **Finite Number of Solutions** Maximum of $\boldsymbol{{}^{n}C_{m}}$ basic feasible solutions exist.

NOTE: Basic variables are variables associated with a BFS, **Non-degenerate BFS** have all basic variables non-zero, while **degenerate BFS** have at least one basic variable equal to zero.

Example

Which of the following vectors

$$\mathbf{x} = [x_1, x_2, x_3, x_4, x_5, x_6, x_7]$$

is a BFS of the system

$$x_1 + 2x_2 + x_3 + 3x_4 + x_5 = 9$$
$$2x_1 + x_2 + 3x_4 + x_6 = 9$$
$$-x_1 + x_2 + x_3 + x_7 = 0$$

$$x_1, x_2, \dots, x_7 \geq 0$$

1) $\mathbf{x}_1 = [2,2,0,1,0,0,0]$
2) $\mathbf{x}_2 = [0,0,9,0,0,9,-9]$
3) $\mathbf{x}_3 = [3,3,0,0,0,0,0]$
4) $\mathbf{x}_4 = [0,0,0,0,9,9,0]$
5) $\mathbf{x}_5 = [1,0,0,0,8,7,1]$
6) $\mathbf{x}_6 = [0,0,0,3,0,0,0]$.

1) Non-zero variable vectors:

$$\alpha_1 = [1,2,-1], \alpha_2 = [2,1,1], \alpha_4 = [3,3,0]$$

Determinant:

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ -1 & 1 & 0 \end{vmatrix} = 0$$

✅ Vectors are **L.D.**

❌ $\mathbf{x}_1$ is **not a BFS.**

2) Non-zero variable vectors:

$$\alpha_3 = [1,0,1], \alpha_6 = [0,1,0], \alpha_7 = [0,0,1]$$

Determinant:

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 1 \neq 0$$

✅ Vectors are **L.I.**

$\mathbf{x}_2$ is a **basic solution**

❌ Not **feasible** as $x_7 < 0$.

3)

Contains **5 zero variables** → may be BFS.

Non-zero vectors:

$$\alpha_1 = [1,2,-1], \alpha_2 = [2,1,1], \alpha_5 = [1,0,0]$$

Determinant:

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ -1 & 1 & 0 \end{vmatrix} = 3 \neq 0$$

✅ L.I. → $\mathbf{x}_3$ is a **B.F.S.**

Basic variables: x_1, x_2, x_5

❗ Degenerate B.F.S. ($x_5 = 0$).

4)

Contains **5 zero variables** → may be B.F.S.

Non-zero vectors:
$\alpha_5 = [1,0,0]$, $\alpha_6 = [0,1,0]$, $\alpha_7 = [0,0,1]$

✅ Vectors are **L.I.** → $\mathbf{x}_4$ is a **B.F.S.**

Basic variables: x_5, x_6, x_7

❗ Degenerate B.F.S. (as $x_7 = 0$).

5)

Does **not contain at least 4 zero variables** or contains **> 3 non-zero variables**

❌ $\mathbf{x}_5$ is **not a BFS**.

6)

One non-zero variable:

$$\alpha_4 = [3,3,0]$$

Check with α_6, α_7:

$$\begin{vmatrix} 3 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 3 \neq 0$$

✅ L.I. → $\mathbf{x}_6$ is a **B.F.S.**

Basic variables: x_4, x_6, x_7

❗ Degenerate B.F.S. (as $x_6 = x_7 = 0$)

Slack and Surplus variables: Convert Inequalities to Equalities

Slack Variables

Slack variables are **positive variables added to the left-hand side** of $\leq$ constraints to create equalities.

Example Problem

$$Maximize\ Z = 3x_1 + 2x_2$$

Subject to

$$x_1 + x_2 \leq 4$$
$$x_1 - x_2 \leq 2$$

Converted Form

$$x_1 + x_2 + s_1 = 4$$
$$x_1 - x_2 + s_2 = 2$$

where $s_1, s_2 \geq 0$.

Here s_1 and s_2 are **slack variables** representing unused resources or capacity.

Surplus Variables

Surplus variables are positive variables **subtracted from the left-hand side** of $\geq$ constraints to create equalities.

Example Transformation

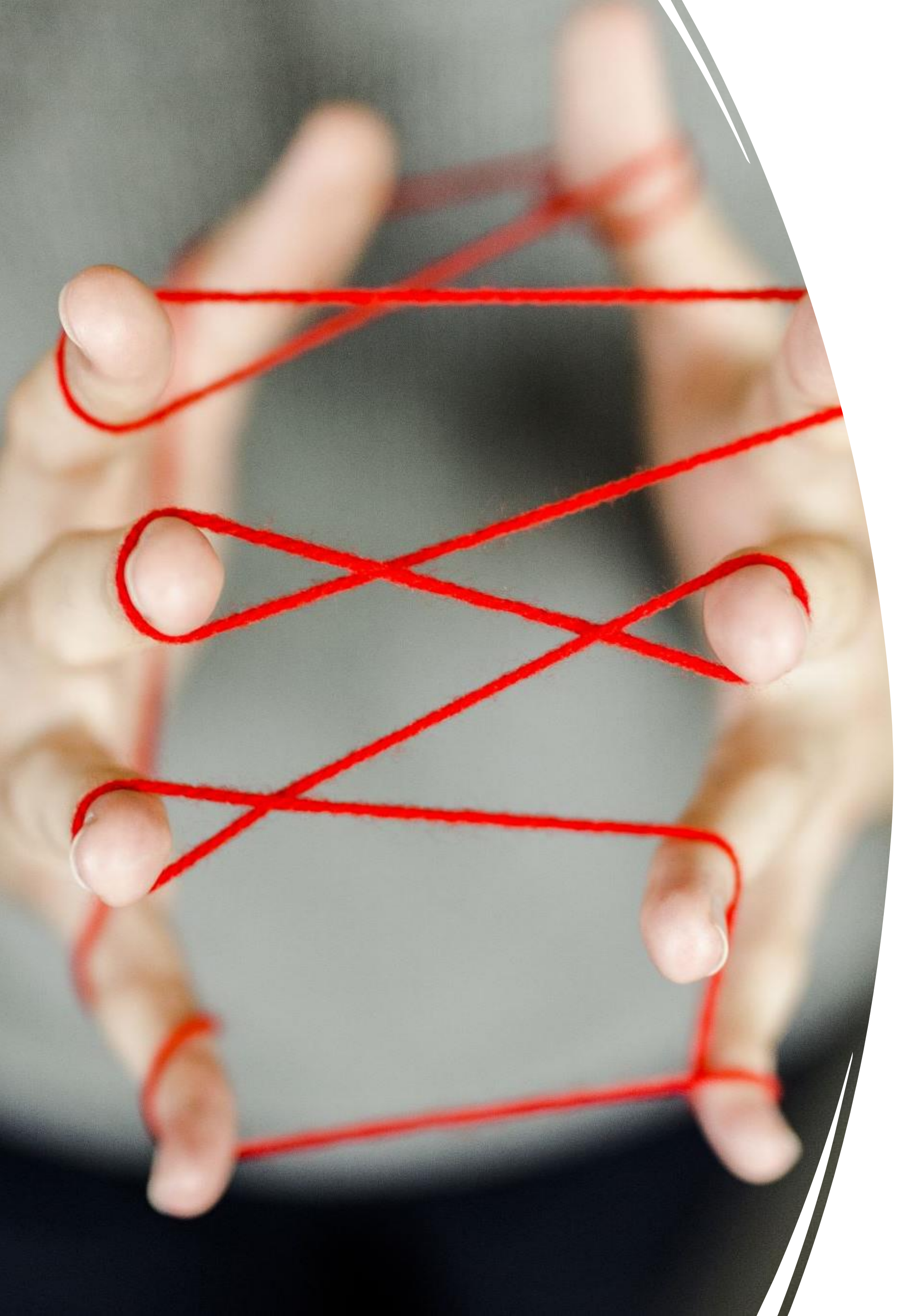
For constraints like

$$2x_1 + x_2 \geq 10,$$

we subtract surplus variable t_1 to obtain

$$2x_1 + x_2 - t_1 = 10,$$

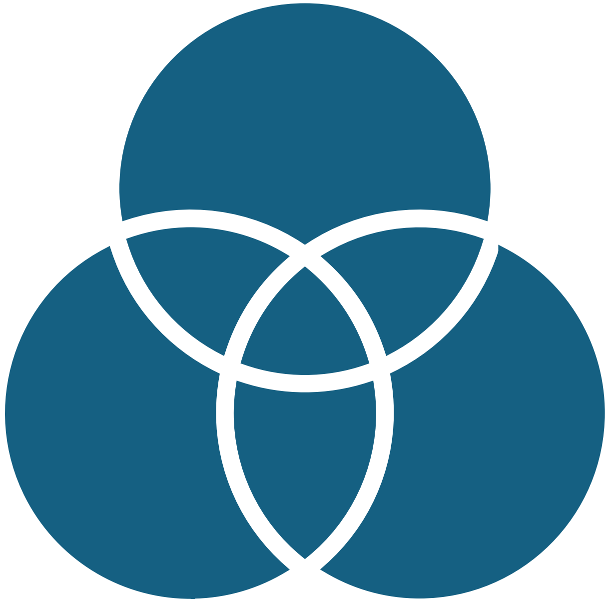
where $t_1 \geq 0$.



Key Distinction

- .Slack variables represent unused capacity in $\leq$ constraints, while surplus variables represent excess production or resources beyond minimum requirements in $\geq$ constraints.

Both are essential for converting inequality constraints into the equality form required by the simplex method.



Convex Sets and Their Properties

An essential foundation for linear programming and optimization theory

Convex Combinations

Definition

A **convex combination** of points $x_1, x_2, \dots, x_n$ is

$$x = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n$$

where $\lambda_i \geq 0$ for all i and $\sum_{i=1}^{n} \lambda_i = 1$.

For two points, this simplifies to

$$x = \lambda x_1 + (1 - \lambda) x_2$$

where $0 \leq \lambda \leq 1$.

Key insight: The line segment joining x_1 and x_2 is precisely the set of all convex combinations of these two points.

What Makes a Set Convex?

A set is **convex** if for any two points in the set, the entire line segment joining them also lies in the set.

Mathematically: if $x_1, x_2 \in S$, then $\lambda x_1 + (1 - \lambda)x_2 \in S$,
for all $0 \leq \lambda \leq 1$

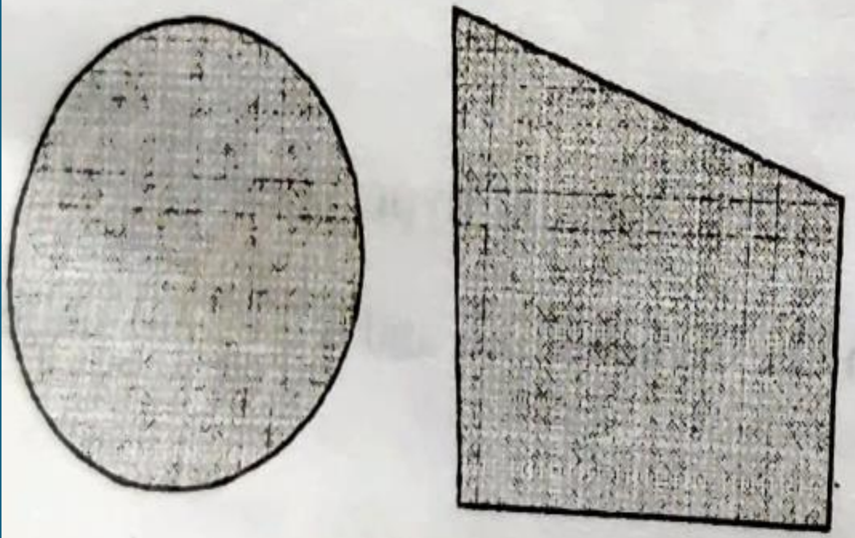
Convex Example

A circle and its interior: $\{(x_1, x_2): x_1^2 + x_2^2 \leq 1\}$

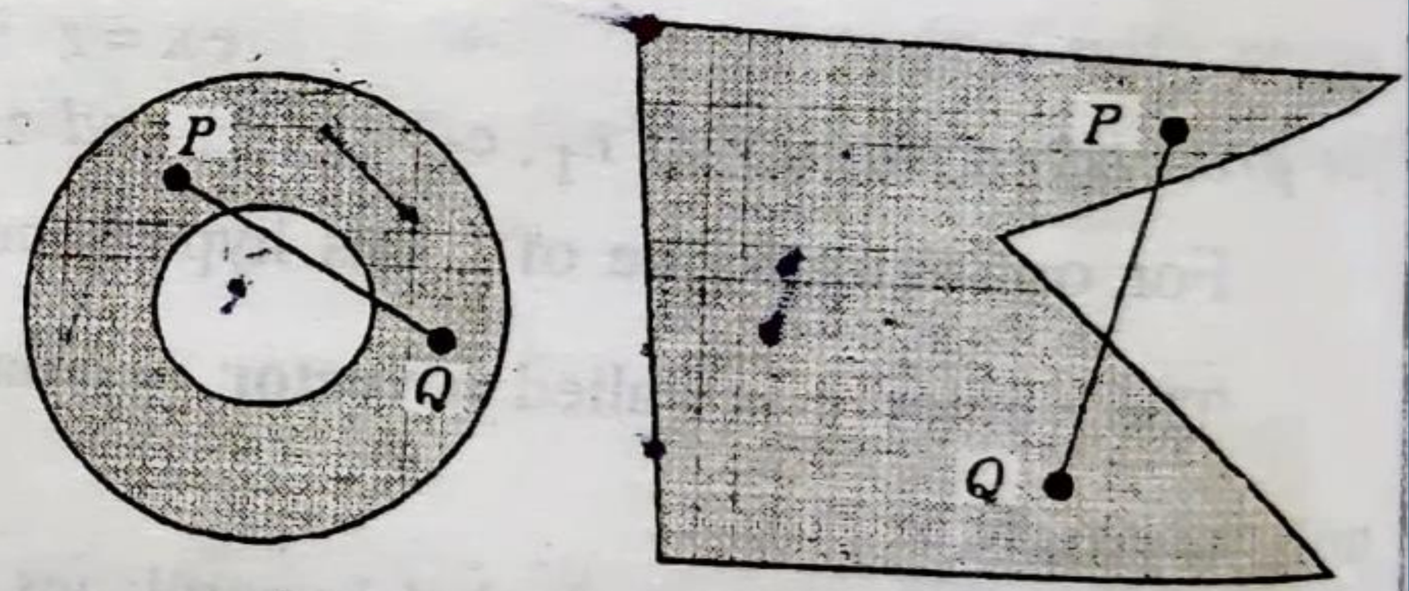
Non-Convex Example

An annulus (ring): $\{(x_1, x_2): 1 \leq x_1^2 + x_2^2 \leq 4\}$

- A set of one point is always convex.



Convex Sets



Non Convex Sets

Extreme Points and Convex Hulls

Extreme Points of a Convex Set

A point in a convex set is called an **extreme point** if it cannot be expressed as a convex combination of two distinct points in C.

Mathematically, x is an extreme point if there **do not exist** $x_1 \neq x_2$ in C such that

$$x = \lambda x_1 + (1 - \lambda) x_2 \text{ for } 0 \leq \lambda \leq 1.$$

Extreme points are always boundary points, but not all boundary points are extreme points.

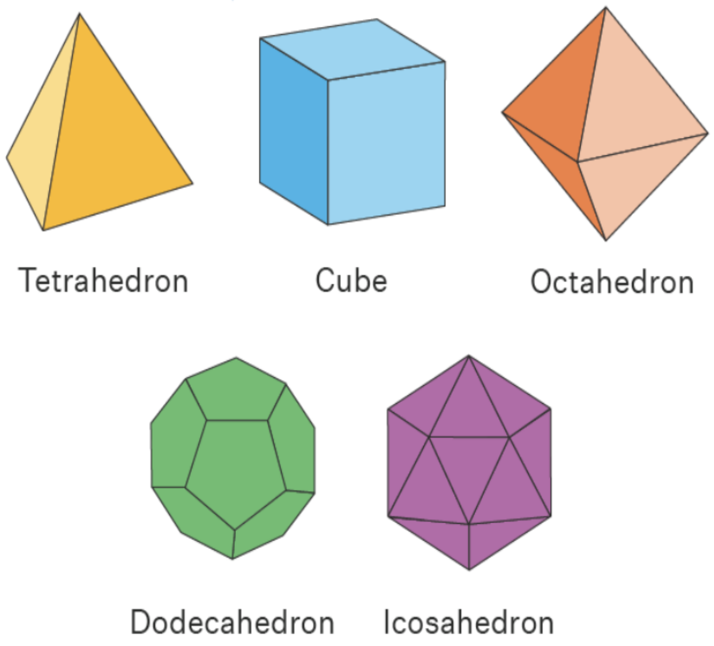
If the point $x \in C$ is not extreme then it is called an **internal point.**

Convex Hull

The **convex hull** $\mathrm{C}(\mathrm{X})$ of a set X is the smallest convex set containing X— formed by taking all convex combinations of points from X.

Example: The convex hull of a cube's eight vertices is the entire cube, including its interior.

Convex Functions and Convex Polyhedron



Strictly Convex Functions

- A function $f(x)$ is said to be **strictly convex** at x if for any two other distinct points x_1 and x_2

$$f\{\lambda x_1 + (1-\lambda)x_2\} < \lambda f(x_1) + (1-\lambda) f(x_2),$$

where $0 \leq \lambda \leq 1$.

- If $-f(x)$ is strictly convex then $f(x)$ is called **strictly concave function.**

Convex Polyhedron

The set of all possible convex combinations of finite number of points.

Example: The set of the area of a triangle is a convex polyhedron of the set of its vertices.

Fundamental Results

- **Hyperplanes are Convex**

  Any hyperplane $\{x\colon\ cx = z\}$ is a convex set.

- **Closed Half-Spaces are Convex**

  Both closed half-spaces $\{x\colon\ cx \geq z\}$ and $\{x\colon\ cx \leq z\}$ are convex sets.

- **Intersection Preserves Convexity**

  The intersection of any collection of convex sets is convex.

- **Convex Combinations Generate Convex Sets**

  The set of all convex combinations of finitely many points forms a convex set.

Linear Programming Connection

The Feasibility Set of Solutions is Convex.

For an LPP

$$A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0,$$

the (non empty) set of all feasible solutions is a convex set.

Proof Sketch

If $\mathbf{x}_1$ and $\mathbf{x}_2$ are feasible solution, then

$\mathbf{x}_3 = \lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2$ with $0 \leq \lambda \leq 1$ satisfies

- $\mathrm{A}\mathbf{x}_3 = \lambda\mathrm{A}\mathbf{x}_1 + (1-\lambda)\mathrm{A}\mathbf{x}_2 = \lambda\mathbf{b} + (1-\lambda)\mathbf{b} = \mathbf{b}$
- $\mathbf{x}_3 = \lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2 \geq 0$ (since all components are non-negative)

Therefore $\mathbf{x}_3$ is feasible solution, proving convexity.

The Fundamental Results of Linear Programming

- **Basic Feasible Solutions $\leftrightarrow$ Extreme Points**

Every basic feasible solution (BFS) of $A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0$, corresponds to an extreme point of the convex feasible region, and vice versa.

- **Finite Extreme Points**

The extreme points of the convex set of feasible solutions are finite in number.

- **Multiple Optimal Solutions**

If the objective function of an LPP attains its optimal value at two extreme points, then it has infinitely many optimal values (all convex combinations).

Optimality: If the feasible region is a convex polyhedron with a bounded optimal solution, at least one extreme point provides the optimal value.

Example

Show that $S = \{(x_1, x_2, x_3): 2x_1 - x_2 + x_3 \leq 4\} \subset R^3$, is a convex set.

For any $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in S$, then

$$2x_1 - x_2 + x_3 \leq 4, \quad 2y_1 - y_2 + y_3 \leq 4. \qquad (1)$$

Let $w = (w_1, w_2, w_3) = \lambda x + (1 - \lambda)y, 0 \leq \lambda \leq\ 1.$ Then,

$$\begin{aligned} 2w_1 - w_2 + w_3 \quad &= \quad \lambda(2x_1 - x_2 + x_3) + (1 - \lambda)(2y_1 - y_2 + y_3) \\ &\leq \ 4\lambda + 4(1 - \lambda) = 4. \end{aligned}$$

$\Rightarrow$ w $\in S$. Hence, S is a convex set.

Example

Find all the basic feasible solutions for the equations

$$2x_1 + 6x_2 + 2x_3 + x_4 = 3$$
$$6x_1 + 4x_2 + 4x_3 + 6x_4 = 2$$
$$x_i \geq 0$$

and determine the associated general convex combination of the extreme point solution.

Solution

I. The given system can be written as

$$A\mathbf{x} = \mathbf{b}$$

Where

$A = (\alpha_1, \alpha_2, \alpha_3, \alpha_4), \mathbf{x} = (x_1, x_2, x_3, x_4)', \ \mathbf{b} = (3,2)'$

with

$$\alpha_1 = \begin{pmatrix}2\\6\end{pmatrix}, \alpha_2 = \begin{pmatrix}6\\4\end{pmatrix}, \alpha_3 = \begin{pmatrix}2\\4\end{pmatrix}, \alpha_4 = \begin{pmatrix}1\\6\end{pmatrix}.$$

Since there are 4 variables and 2 equations, **at most** ${}^4C_2 = 6$ basic solutions can exist.

II. Linear Independence

$$B_1 = [\alpha_1, \alpha_2] = \begin{bmatrix}2 & 6\\6 & 4\end{bmatrix}, B_2 = [\alpha_1, \alpha_3] = \begin{bmatrix}2 & 2\\6 & 4\end{bmatrix}$$

$$B_3 = [\alpha_1, \alpha_4] = \begin{bmatrix}2 & 1\\6 & 6\end{bmatrix}, B_4 = [\alpha_2, \alpha_3] = \begin{bmatrix}6 & 2\\4 & 4\end{bmatrix}$$

$$B_5 = [\alpha_2, \alpha_4] = \begin{bmatrix}6 & 1\\4 & 6\end{bmatrix}, B_6 = [\alpha_3, \alpha_4] = \begin{bmatrix}2 & 1\\4 & 6\end{bmatrix}$$

All 2×2 submatrices B_i formed from pairs of columns have non-zero determinants.

Hence, all **six basic solutions exist.**

III. Basic Feasible Solutions (BFS)

Let $\mathbf{x}_{B_i} = B_i^{-1}\mathbf{b}$, then

$\mathbf{x}_{B_1} = (x_1, x_2) = (0, 1/2),$

$\mathbf{x}_{B_2} = (x_1, x_3) = (-2, 7/2),$

$\mathbf{x}_{B_3} = (x_1, x_4) = (8/3, -7/3),$

$\mathbf{x}_{B_4} = (x_2, x_3) = (1/2, 0),$

$\mathbf{x}_{B_5} = (x_2, x_4) = (1/2, 0),$

$\mathbf{x}_{B_6} = (x_3, x_4) = (2, -1)$

Solving $B_i\mathbf{x} = \mathbf{b}$,only **three basic solutions are non-negative**, hence feasible.

All these BFS coincide at $\mathbf{x} = (0, 1/2, 0, 0)$.

Conclusion

BFS correspond to **extreme points**

All BFS are identical. $\Rightarrow$ $\boxed{\textit{Unique extreme point solution}}$