Part IV

Lecture: Some Special Continuous Probability Distributions

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- Probability density function.
- Requirements for a valid probability density function.
- Probability distribution function.

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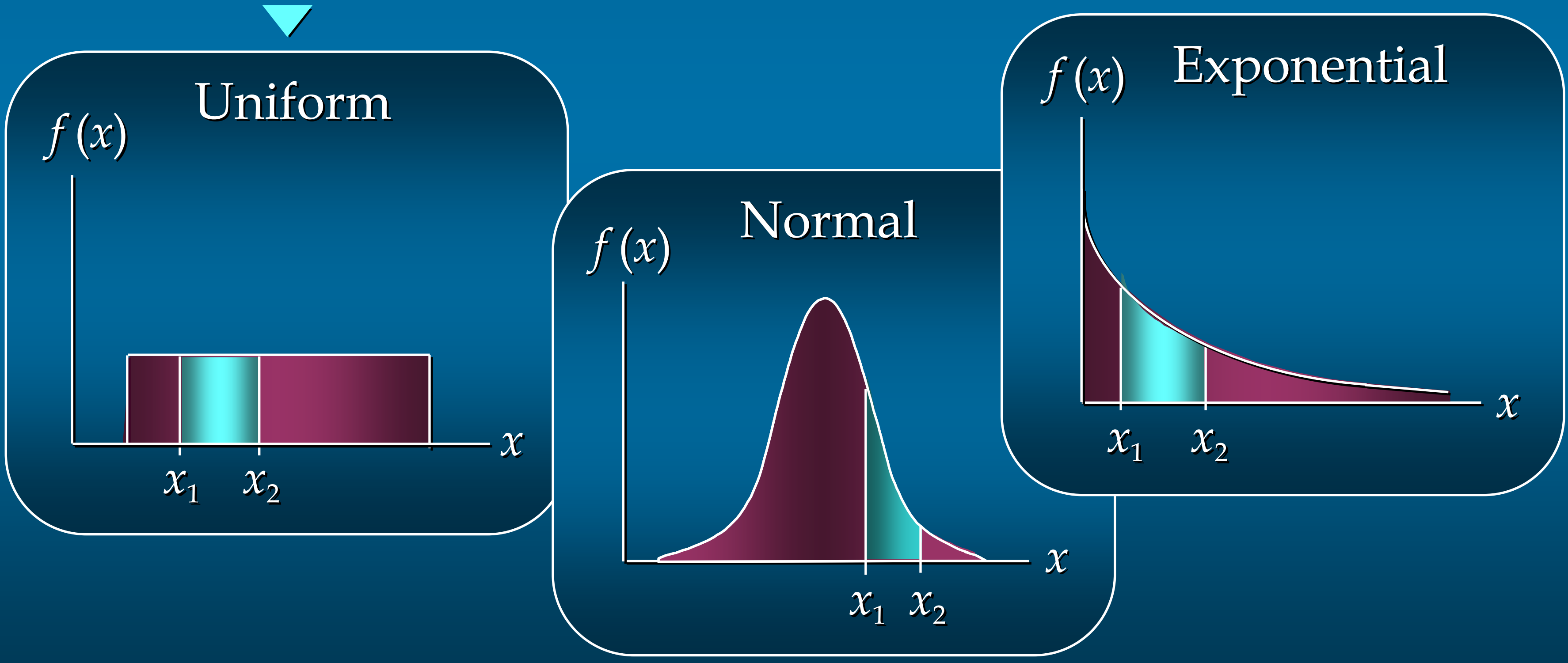
- The height of a continuous probability density function is not a probability.
- The probability of the random variable taking on a value within some given interval is defined to be the area under the graph of the pdf in the interval.
- The probability that a continuous random variable takes on any particular value is zero.

Continuous Probability Distributions

- The probability of the random variable assuming a value within some given interval from x_1 to x_2 is defined to be the area under the graph of the probability density function between x_1 and x_2.



Uniform (or Rectangular) Distribution:

Definition

A random variable X *is said to have a uniform distribution over an interval* $[a, b], -\infty < a < b < \infty$*, if its pdf is given by*

$$f(x) = \begin{cases} \frac{1}{b-a}, & a \leqslant x \leqslant b \\ 0, & otherwise. \end{cases}$$

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(ii) The distribution is also known as rectangular distribution, since the curve $y = f(x)$ describes a rectangle over the $x-$ axis and between the ordinates at $x = a$ and $x = b$.
The CDF of X is given by

$$F(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \leqslant x < b \\ 1, & x \geqslant b. \end{cases}$$

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Subway trains on a certain line run every half-hour between mid-night and six in the morning. What is the probability that a man entering the station at a random time during this period will have to wait at least twenty minutes?

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Solution: $f(x) = 1/30,\ 0 < x < 30.\ P(x \geqslant 20) = \int_{20}^{30} f(x)dx = 1/3.$

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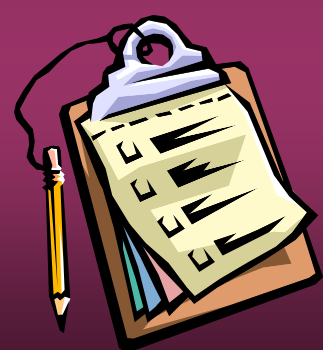
Exponential Probability Distribution

- The exponential probability distribution is useful in describing the time it takes to complete a task.
- The exponential random variables can be used to describe:

Time between vehicle arrivals at a toll booth

Time required to complete a questionnaire



Distance between major defects in a highway

Probability density function

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geqslant 0 \\ 0, & x < 0, \end{cases}$$

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N.B.: $M_X(t) = \frac{\lambda}{\lambda-1} = (1 - \frac{t}{\lambda})^{-1} = \sum_{r=0}^{\infty} (\frac{t}{\lambda})^r,\ \lambda > t,$

so, $\alpha_k = E(X^k) =$ coefficient of $\frac{t^k}{k!}$ in $M_X(t) = \frac{k!}{\lambda^k},\ k = 1, 2, ...$

Hence, mean$= \alpha_1 = 1/\lambda$, $\alpha_2 = 2/\lambda^2 \Rightarrow Var(X) = 1/\lambda^2$

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Memoryless Property:

Importance of Normal Distribution

1. Most of the distributions (Binomial, Poisson, etc.) occurring in practice can be approximated by normal distribution. Also many of the sampling distributions (Student's, F, Chi-square, etc.) tend to normality for large samples.

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2. Even if a variable is not normally distributed, it can sometimes be brought to normal form by simple transformation of variable.
3. Normal distribution finds large applications in statistical Quality Control in industry for setting control limits.

Normal Probability Distribution

- It has been used in a wide variety of applications:
  - Heights of people

  - Scientific measurements

Normal Probability Distribution

- It has been used in a wide variety of applications:

  - Test scores

  - Amounts of rainfall

Probability density function

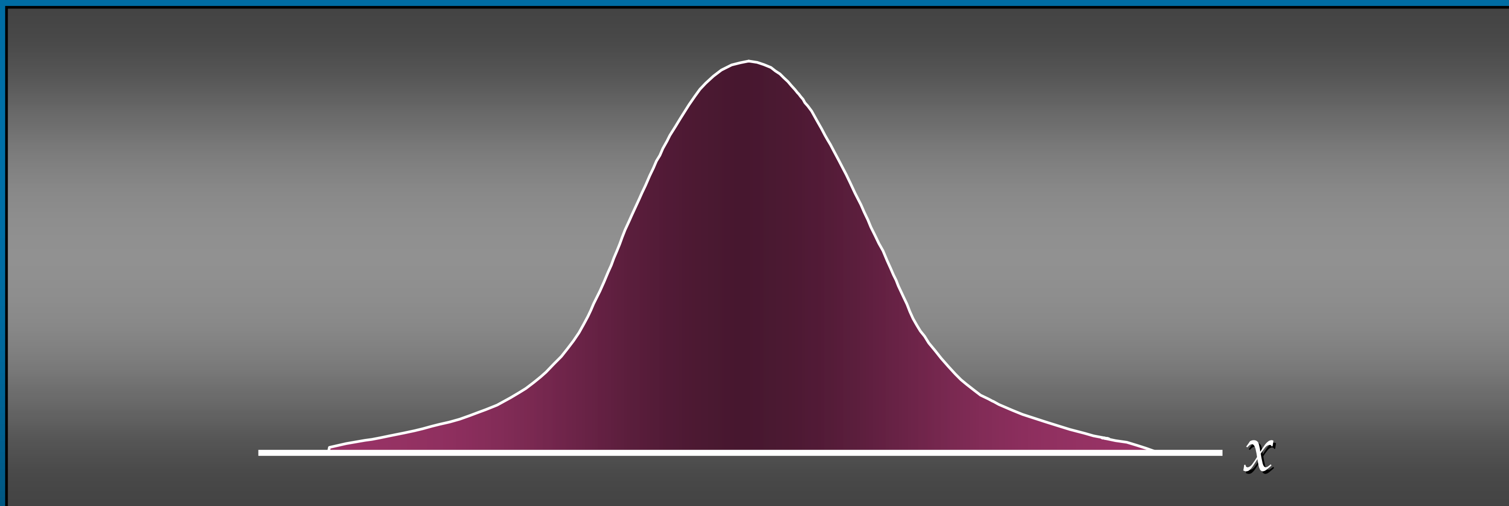
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \; for \; -\infty < x < \infty,$$

where $\mu =$ mean of the random variable
and $\sigma^2 =$ variance of the random variable.

Normal Probability Distribution

- Characteristics

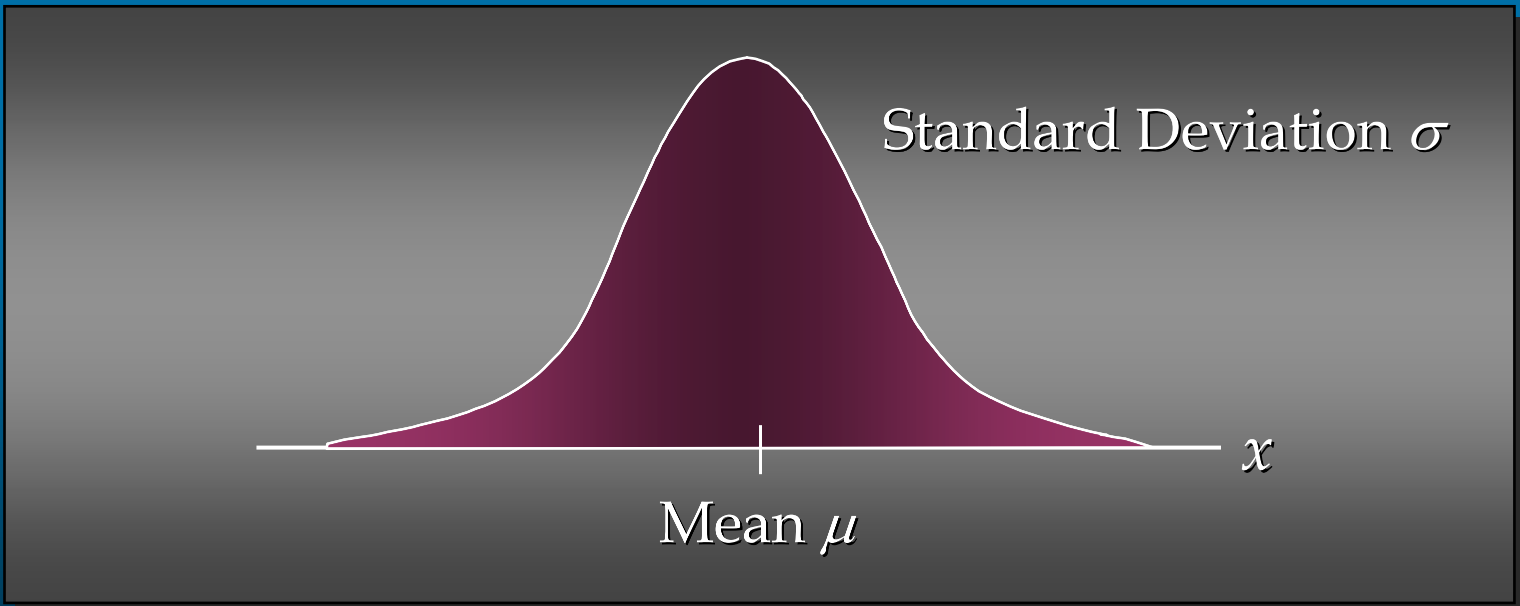
The distribution is symmetric; its skewness measure is zero.



Normal Probability Distribution

- Characteristics

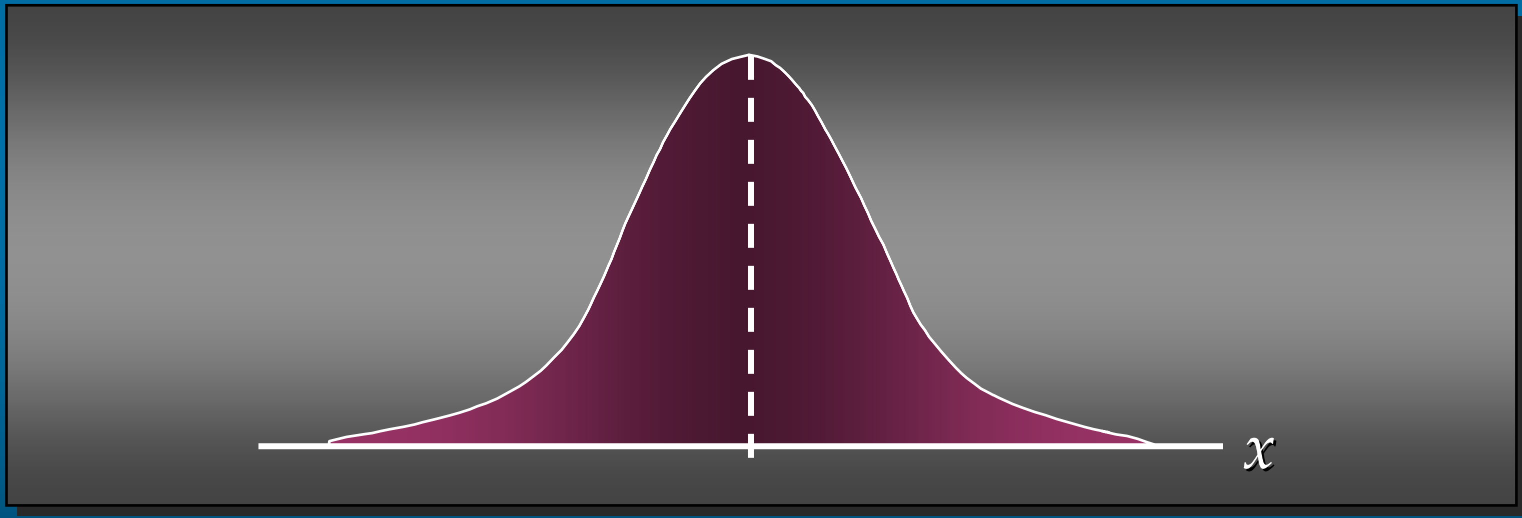
  - The entire family of normal probability distributions is defined by its mean μ and its standard deviation σ.



Normal Probability Distribution

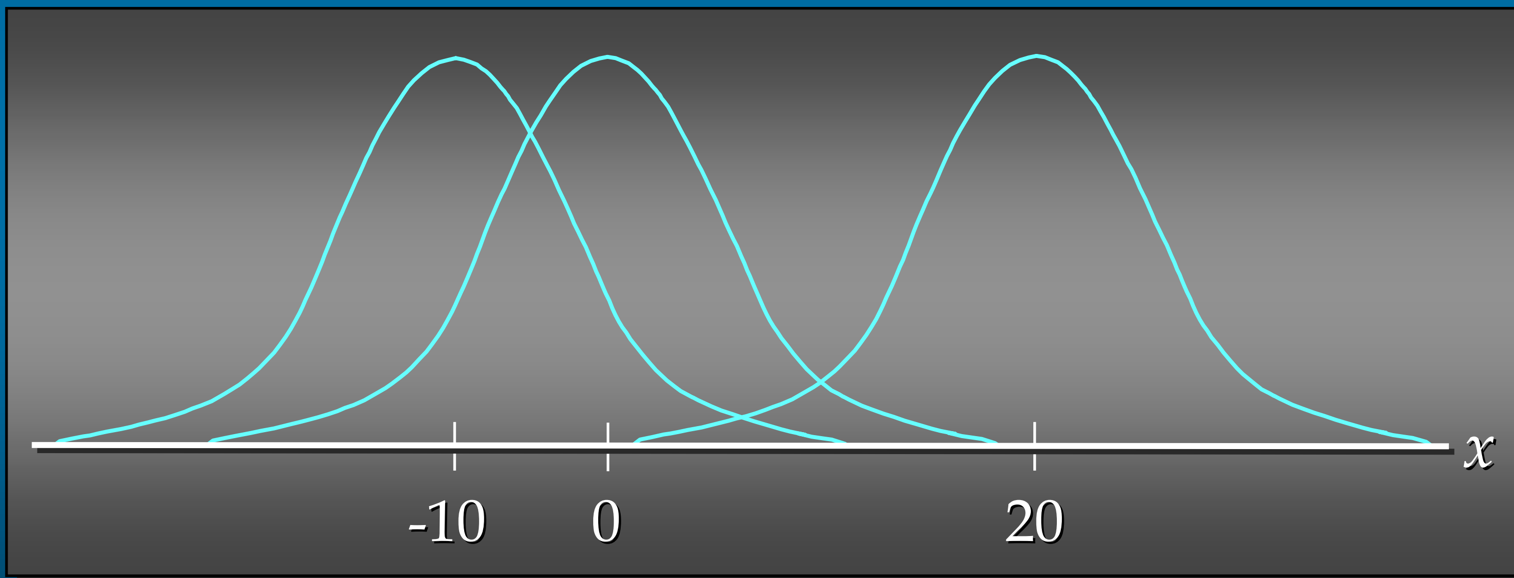
- Characteristics

  - The highest point on the normal curve is at the mean, which is also the median and mode.



Normal Probability Distribution

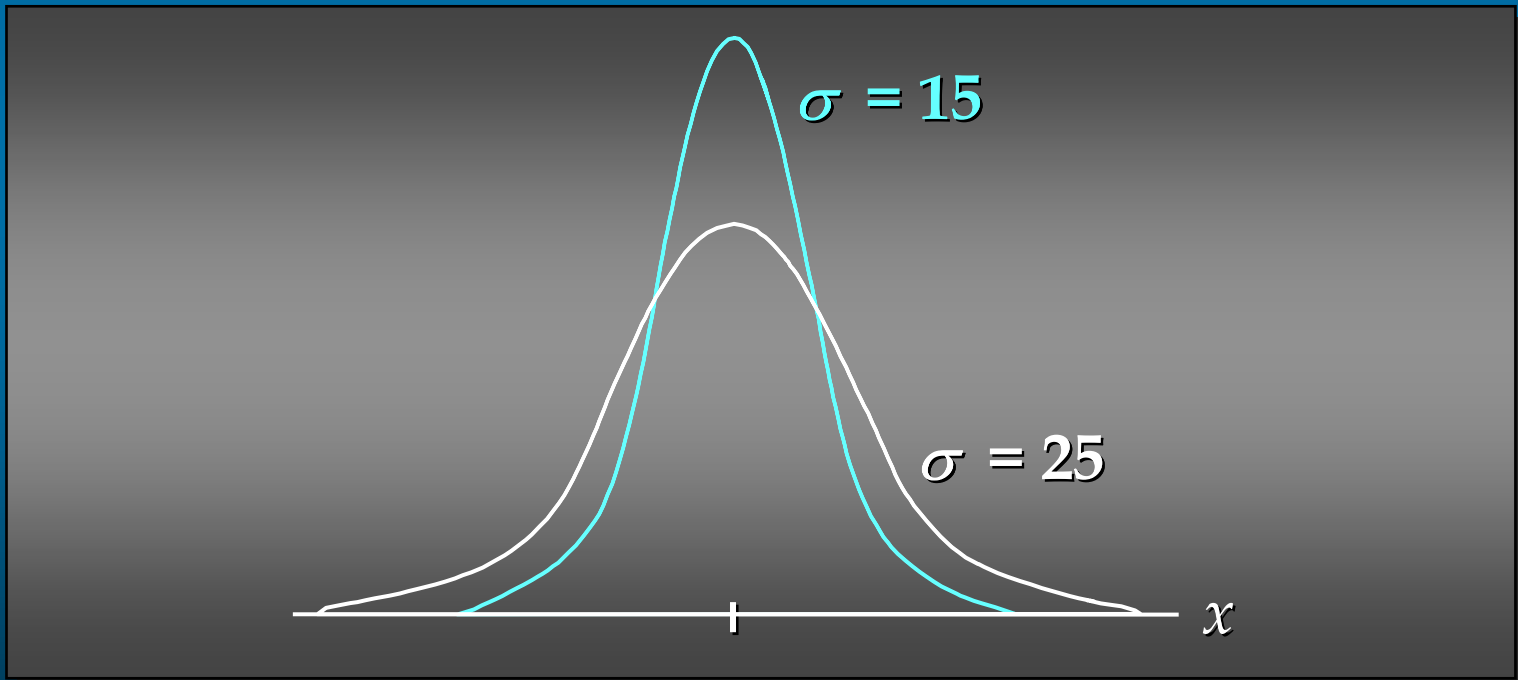
- Characteristics
  - The mean can be any numerical value: negative, zero, or positive.



Normal Probability Distribution

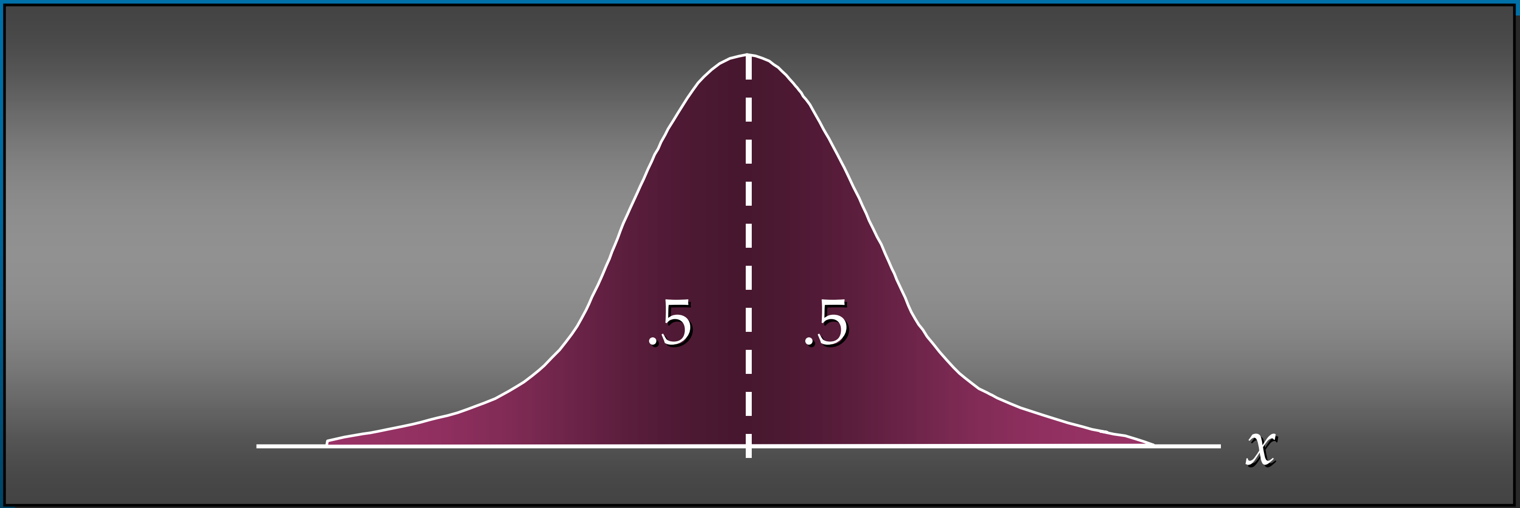
- Characteristics

  - The standard deviation determines the width of the curve: larger values result in wider, flatter curves.



Normal Probability Distribution

- Characteristics
  - Probabilities for the normal random variable are given by <u>areas under the curve</u>. The total area under the curve is 1 (.5 to the left of the mean and .5 to the right).



Normal Probability Distribution

- Characteristics
  - 68.26% of values of a normal random variable are within +/- 1 standard deviation of its mean.
  - 95.44% of values of a normal random variable are within +/- 2 standard deviations of its mean.
  - 99.72% of values of a normal random variable are within +/- 3 standard deviations of its mean.

Normal Probability Distribution

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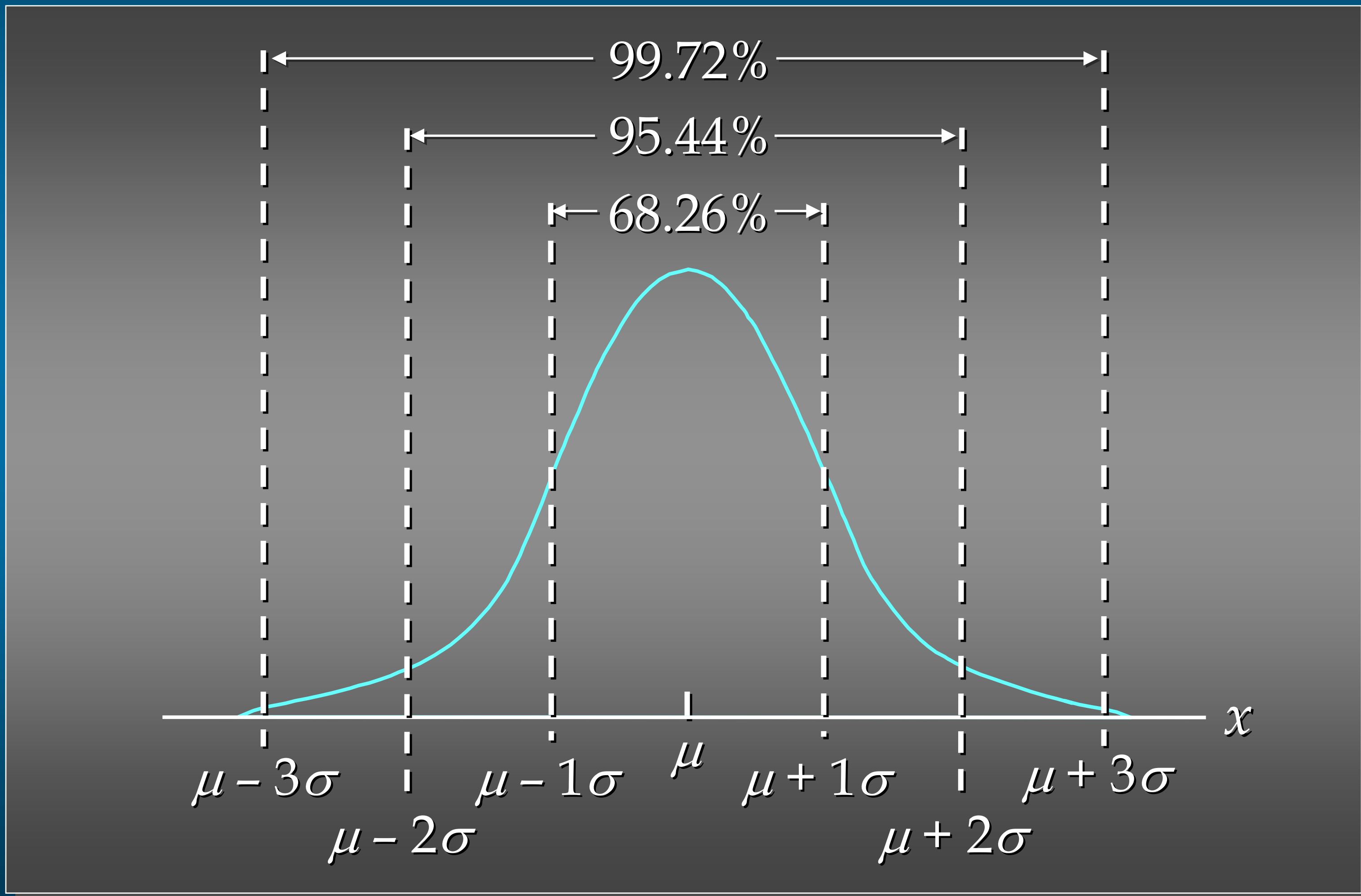

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where $\mu =$ mean of the random variable
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Standard Normal/Gaussian Distribution

A random variable that has a mean 0 and standard deviation 1. The normal random variable X with mean μ and standard deviation σ can be converted to standard normal random variable by

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$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \; -\infty < z < \infty$$

and the corresponding *CDF* is

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2} du.$$

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Result

$$P(a \leqslant X \leqslant b) = P\left(\frac{a-\mu}{\sigma} \leqslant Z \leqslant \frac{b-\mu}{\sigma}\right) = P\left(Z \leqslant \frac{b-\mu}{\sigma}\right) - P\left(Z \leqslant \frac{a-\mu}{\sigma}\right) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

Grear Tire Problem

Suppose that the Grear Tire Company just developed a new steel-belted radial tire that will be sold through a national chain of discount stores. Because the tire is a new product, Grear's Management believes that the mileage guarantee offered with the tire will be an important factor in the consumer acceptance of the product. Before finalizing the tire mileage guarantee policy, Grear's management wants some probability information concerning the number of miles the tires will last.

From actual road tests with the tires, Grear's engineering group estimates the mean tire mileage at 36500 miles and standard deviation 5000 miles. What percentage of the tire, then, can be expected to last more than 40000 miles?

Normal Distribution as an approximation of Binomial Distribution

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$$(mean(\mu) = np\ and\ standard\ deviation(\sigma) = \sqrt{npq})$$

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Exercise

(i) On the basis of past experience, automobile inspectors in Pennsylvania have noticed that 5 percent of all cars coming in for their annual inspection fail to pass. Using the normal approximation to the binomial, find the probability that between 7 and 18 of the next 200 cars to enter the Lancaster inspection station will fail the inspection.

(ii) Assume that in a digital communication channel, the no. of bits received in error can be modeled by binomial distribution. The prob. that a bit is received in error is 10^{-5}. If 16 million bits are transmitted, what is the prob. that 150 or fewer errors occur?

Log-normal Distribution

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If $X \sim N(\mu, \sigma^2)$, then $Y = e^X$ is a Log-normal random variable.

The mean and variance of Log-normal distribution are
$E(X) = e^{\mu+\sigma^2/2}$ and $V(X) = e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right)$.

Example

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Solution

$$\begin{aligned} P(X > 10,000) &= 1 - P\left(\frac{\ln X - \mu}{\sigma} \leqslant \frac{\ln(10,000) - 10}{1.5}\right) \\ &= 1 - \Phi(-0.52) = 0.70 \end{aligned}$$

The question is to determine x such that $P(X > x) = 0.99$. Therefore,

$$P(X > x) = 1 - P\left(\frac{\ln X - \mu}{\sigma} \leqslant \frac{\ln x - 10}{1.5}\right) = 0.99$$

From table, $1 - \Phi(z) = 0.99$ when $z = -2.33$. This gives $x = exp(6.506)$ i.e., $x = 668.48$ hrs.

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THANK YOU