# Part III

## Lecture: Some Special Discrete Probability Distributions

# Prerequisite: Discrete Random Variables

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1. Example.
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3. Probability distribution function.
4. Requirements for a valid discrete probability distribution.

## Exercise

The following table shows a partial probability distribution for the MRA Company's projected profits (in thousands of dollars) for the first year of operation (the negative value denotes a loss). Find the missing value of $f(200)$. What is the probability that MRA will be profitable? What is the probability that MRA will make at least $100,000?

| $x$ | $f(x)$ |
|---|---|
| -100 | 0.10 |
| 0 | 0.20 |
| 50 | 0.30 |
| 100 | 0.25 |
| 150 | 0.10 |
| 200 | $*$ |

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- Conditions 2, 3, and 4 $\Rightarrow$ **Bernoulli trials**
- In addition, condition 1 $\Rightarrow$ **Binomial experiment**

## Binomial Distribution: James Bernoulli (1654-1705)

$$f(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x},\ x = 0, 1, \ldots, n$$

where

$$n = number\ of\ trials$$

$$p = probability\ of\ success\ in\ one\ trial$$

$$x = number\ of\ success\ in\ n\ trials$$

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- **Mean:** $\mu = np$.
- **Variance:** $\sigma^2 = np(1-p)$.
- **MGF:** $M_X(t) = E(e^{tX}) = \sum_{x=0}^{n} e^{tx}\binom{n}{x}p^x q^{n-x} = \sum_{x=0}^{n}\binom{n}{x}(pe^t)^x q^{n-x} = (q+pe^t)^n$.

# Binomial Distribution

- Binomial Probability Function

$$f(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{(n-x)}$$

$$\frac{n!}{x!(n-x)!}$$

Number of experimental outcomes providing exactly $x$ successes in $n$ trials

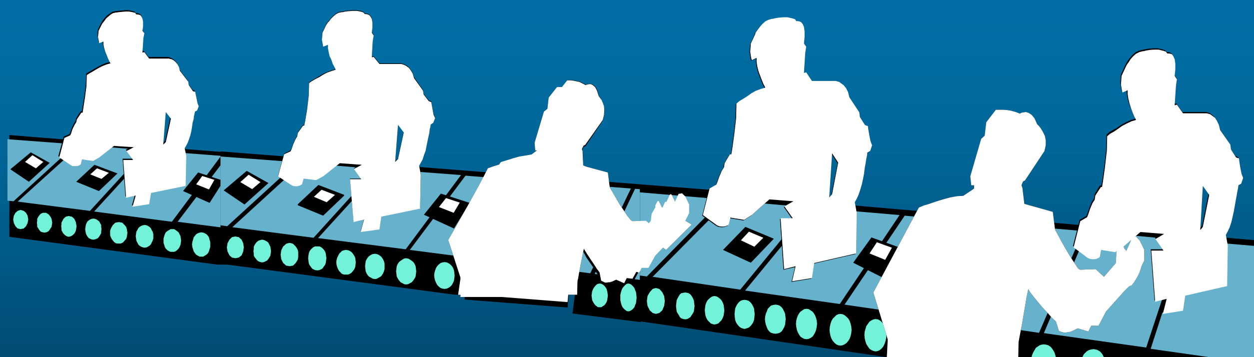
$$p^x (1-p)^{(n-x)}$$

Probability of a particular sequence of trial outcomes with x successes in $n$ trials

# Binomial Distribution

- Example: Evans Electronics

  Evans is concerned about a low retention rate for employees. In recent years, management has seen a turnover of 10% of the hourly employees annually. Thus, for any hourly employee chosen at random, management estimates a probability of 0.1 that the person will not be with the company next year.



# Binomial Distribution

- Using the Binomial Probability Function

Choosing 3 hourly employees at random, what is the probability that 1 of them will leave the company this year?

Let: $p = .10,\ n = 3,\ x = 1$

$$f(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{(n-x)}$$

$$f(1) = \frac{3!}{1!(3-1)!}(0.1)^1(0.9)^2 = 3(.1)(.81) = .243$$

# Binomial Distribution

- Expected Value

$$E(x) = \mu = 3(.1) = .3 \text{ employees out of } 3$$

- Variance

$$\text{Var}(x) = \sigma^2 = 3(.1)(.9) = .27$$

- Standard Deviation

$$\sigma = \sqrt{3(.1)(.9)} = .52 \text{ employees}$$

## Exercise

Consider the experiment of customers entering the Bata store. If, based on the experience of 3 customers, the store manager estimates that the probability of a customer making a purchase is 0.3. What is the probability that exactly two of the next three customers make a purchase? Find the expected number of customers making a purchase. Also find the variance and standard deviation.

Suppose that during the next month Bata store expects 1000 customers to enter the store. What is the expected number of customers who will make a purchase?

## Geometric Distribution

In a series of Bernoulli trials (independent trials with constant probability $p$ of a success), the random variable $X$ that equals the number of trials until the first success is a geometric random variable with parameter $0 < p < 1$ and

$$P(X = x) = f(x) = (1-p)^{x-1}p, \ x = 1, 2, 3, \ldots, \infty.$$

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- **Mean:** $\mu = 1/p$.
- **Variance:** $\sigma^2 = (1-p)/p^2$.
- **MGF:** $M_X(t) = E(e^{tX}) = pe^t/(1-qe^t)$.

## Negative Binomial Distribution

In a series of Bernoulli trials (independent trials with constant probability $p$ of a success), the random variable $X$ that equals the number of trials until $r$ successes occur is a negative binomial random variable with parameters $p \in (0, 1)$, $r = 1, 2, 3, \ldots$ and

$$P(X = x) = f(x) = \binom{x-1}{r-1}(1-p)^{x-r}p^r, \; x = r, r+1, r+2, \ldots, \infty.$$

Here $f(x)$ represents that $(r-1)$ successes occur in the first $(x-1)$ trials and the $r^{th}$ success occurs on trial $x$.

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## Digital Transmission Example Continue...

Here $P(X = 10) = \binom{9}{3}(0.9)^6(0.1)^4$ is the probability three errors occur in the first nine trials and trial 10 results in the fourth error

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- **Mean:** $\mu = r/p$.
- **Variance:** $\sigma^2 = r(1-p)/p^2$.

# Poisson Distribution: Simeon Denis Poisson (1781-1840)

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Assumptions:

• The probability of an occurrence of the event is the same for any two intervals of equal length.

• The occurrence or nonoccurrence of the event in any interval is independent of the occurrence or nonoccurrence in any other interval.

## Probability function

$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!},\ for\ x = 0, 1, 2, \ldots$$

where

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## Moments:

(i) Mean $\mu = E(X) = \sum_{i=0}^{\infty} i \frac{e^{-\lambda}\lambda^i}{i!} = e^{-\lambda} \sum_{i=1}^{\infty} \frac{\lambda^i}{(i-1)!} = \lambda e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = \lambda e^{-\lambda} e^{\lambda} = \lambda$

(ii) $E[X(X-1)] = \sum_{i=0}^{\infty} i(i-1)e^{-\lambda} \frac{\lambda^i}{i!} = \lambda^2 e^{-\lambda} \sum_{i=2}^{\infty} \frac{\lambda^{i-2}}{(i-2)!} = \lambda^2$

Hence, $\sigma^2 = \lambda^2 - \lambda(\lambda - 1) = \lambda \Rightarrow \sigma = \sqrt{\lambda}$.

(iii) $M_X(t) = \sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!} e^{tx} = \sum_{x=0}^{\infty} \frac{e^{-\lambda}(\lambda e^t)^x}{x!} = e^{\lambda(e^t-1)}$.

# Poisson Distribution

- Example: Mercy Hospital
  - Patients arrive at the emergency room of Mercy Hospital at the average rate of 6 per hour on weekend evenings.
    What is the probability of 4 arrivals in 30 minutes on a weekend evening?

# Poisson Distribution


- Using the Poisson Probability Function

$$\mu = 6/\text{hour} = 3/\text{half-hour}, \quad x = 4$$

$$f(4) = \frac{3^4 (2.71828)^{-3}}{4!} = .1680$$

### Exercise

A radioactive emits on the average 2.5 particles per second. Calculate the probability that two or more particles will be emitted in an interval of 4 seconds.

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Example What is the probability that in a company of 500 employee only one will have B'Day on New Year's day?