

# Algorithms<sup>1</sup>

## 1 Notations

- $G = (V, E)$ 
  - $V$  : Set of vertices/nodes
  - $E$  : Set of edges
- $G.Adj[u]$  : Adjacency list of a vertex  $u$  of the graph  $G$

## 2 All-Pair Shortest Path

Let  $D_{uv}^{(r)}$  be the weight of a shortest path from vertex  $u$  to vertex  $v$  that contains at most  $r$  edges, *i.e.*, weight of a shortest path from vertex  $u$  to vertex  $v$  using  $\leq r$  edges. Thus,  $D_{uv}^{(0)}$  is defined using Equation (1).

$$D_{uv}^{(0)} = \begin{cases} 0 & \text{if } u = v \\ \infty & \text{if } u \neq v \end{cases} \quad (1)$$

$D_{uv}^{(1)}$  is obtained using Equation (2).

$$D_{uv}^{(1)} = \begin{cases} 0 & \text{if } u = v \\ w(u, v) & \text{if } (u, v) \in G.E \\ \infty & \text{otherwise} \end{cases} \quad (2)$$

$D_{uv}^{(r)}$  where  $r \geq 1$  is defined recursively using Equation (3).

$$D_{uv}^{(r)} = \min \left\{ \underbrace{D_{uv}^{(r-1)}}_{\text{Weight of a shortest path from } u \text{ to } v \text{ with atmost } r-1 \text{ edges}}, \min \left\{ \underbrace{D_{uk}^{(r-1)}}_{\text{Weight of a shortest path from } u \text{ to } k \text{ with atmost } r-1 \text{ edges}} + \underbrace{w(k, v)}_{\text{Weight of edge } (k, v)} : k \in \text{All predecessors of } v \right\} \right\} \quad (3)$$

$k$  belongs to the set of vertices for which there is an edge going to  $v$ , *i.e.*, all predecessors of  $v$ . In a complete graph, all the vertices are predecessor of  $v$ . So  $k \in G.V$  in case of complete graph. Considering  $k \in G.V$  also makes the formulation simple. Doing so will not make any change to the shortest path because if there is no edge from  $k$  to  $v$ , the edge weight is  $\infty$  and this  $\infty$  value will not affect the minimum calculation. Thus, the updated formulation is given in Equation (4).

$$\begin{aligned} D_{uv}^{(r)} &= \min \left\{ \underbrace{D_{uv}^{(r-1)}}_{\text{Weight of a shortest path from } u \text{ to } v \text{ with atmost } r-1 \text{ edges}}, \min \left\{ \underbrace{D_{uk}^{(r-1)}}_{\text{Weight of a shortest path from } u \text{ to } k \text{ with atmost } r-1 \text{ edges}} + \underbrace{w(k, v)}_{\text{Weight of edge } (k, v)} : k \in G.V \right\} \right\} \\ &= \min \left\{ \underbrace{D_{uk}^{(r-1)}}_{\text{Weight of a shortest path from } u \text{ to } k \text{ with atmost } r-1 \text{ edges}} + \underbrace{w(k, v)}_{\text{Weight of edge } (k, v)} : k \in G.V \right\} \end{aligned} \quad (4)$$

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<sup>1</sup>Please verify the content.

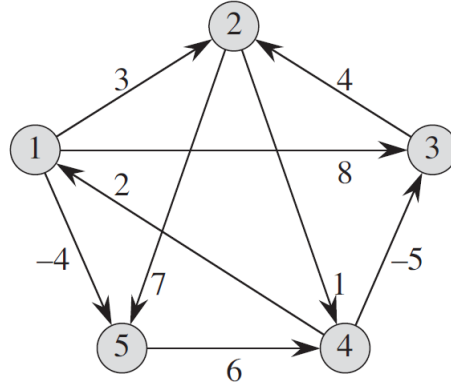


Figure 1: A directed graph.

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**Algorithm 1** APSP-RECURSIVE

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**Input:** A directed graph  $G = (V, E)$

**Output:** Shortest distance between all pair of vertices

- 1:  $n \leftarrow |G.V|$  ▷ Number of vertices in  $G$
  - 2:  $m \leftarrow |G.E|$  ▷ Number of edges in  $G$
  - 3: Create a distance matrix  $D[1, 2, \dots, n][1, 2, \dots, n]$  of size  $n \times n$
  - 4: **for each** vertex  $u \in G.V$  **do**
  - 5:     **for each** vertex  $v \in G.V$  **do**
  - 6:          $D[u][v] \leftarrow$  APSP-HELPER( $n - 1, u, v$ ) ▷ Weight of a shortest path from vertex  $u$  to vertex  $v$  that contains at most  $n - 1$  edges
  - 7:     **end for**
  - 8: **end for**
  - 9: **return**  $D$
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**Algorithm 2** APSP-HELPER( $r, u, v$ )

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- 1: **if**  $r = 0$  **then**
- 2:     **if**  $u = v$  **then**
- 3:         **return** 0 ▷ Weight of a shortest path from vertex  $u$  to vertex  $v$  that contains at most 0 edge is 0 because vertex  $u$  and vertex  $v$  are the same
- 4:     **else**
- 5:         **return**  $\infty$  ▷ Weight of a shortest path from vertex  $u$  to vertex  $v$  that contains at most 0 edge is  $\infty$  because vertex  $u$  and vertex  $v$  are different
- 6:     **end if**
- 7: **else**

Follow the formulation 
$$D_{uv}^{(r)} = \min \left\{ \underbrace{D_{uk}^{(r-1)}}_{\text{Weight of a shortest path from } u \text{ to } k \text{ with atmost } r-1 \text{ edges}} + \underbrace{w(k, v)}_{\text{Weight of edge } (k, v)} : k \in G.V \right\} \quad (5)$$

- 8:     minDistance  $\leftarrow \infty$
  - 9:     **for each** vertex  $k \in G.V$  **do**
  - 10:         dist  $\leftarrow$  APSP-HELPER( $r - 1, u, k$ ) +  $w(k, v)$
  - 11:         **if** dist < minDistance **then**
  - 12:             minDistance  $\leftarrow$  dist
  - 13:         **end if**
  - 14:     **end for**
  - 15:     **return** minDistance
  - 16: **end if**
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**Recurrence Relation to Obtain the Shortest Path From Vertex  $u$  to Vertex  $v$ :** Let  $N = n$

$$\begin{aligned}
T(n-1) &= \underbrace{N}_{\text{No. of times APSP-HELPER is called in line 10}} \left[ \underbrace{T(n-2)}_{\text{Time by APSP-HELPER in line 10}} + \underbrace{\mathcal{O}(1)}_{\text{Weight addition in line 10}} \right] + \mathcal{O}(1) \\
&= NT(n-2) + \mathcal{O}(N) \\
&= NT(n-2) + N \\
&= N[NT(n-3) + N] + N \\
&= N^2T(n-3) + N^2 + N \\
&= N^2[NT(n-4) + N] + N^2 + N \\
&= N^3T(n-4) + N^3 + N^2 + N \\
&= N^3[NT(n-5) + N] + N^3 + N^2 + N \\
&= N^4T(n-5) + N^4 + N^3 + N^2 + N \\
&= \vdots \\
&= N^{N-1}T(n-n) + N^{N-1} + N^{N-2} + \dots + N^4 + N^3 + N^2 + N \\
&= N^{N-1}T(0) + N^{N-1} + N^{N-2} + \dots + N^4 + N^3 + N^2 + N \\
&= N^{N-1} + N^{N-1} + N^{N-2} + \dots + N^4 + N^3 + N^2 + N \\
&= N^{N-1} + N^{N-1} + N^{N-2} + \dots + N^4 + N^3 + N^2 + N \\
&= N^{N-1} + N \frac{N^{N-1} - 1}{N - 1} \\
&= \mathcal{O}(N^{N-1}) \\
&= \mathcal{O}(n^{n-1}) \tag{6}
\end{aligned}$$

- Time complexity to obtain the shortest path from vertex  $u$  to vertex  $v$ 
  - $T(n) = \mathcal{O}(n!)$
- Need to find the shortest path between each pair of vertices
- Time complexity to obtain the shortest path between each pair of vertices
  - Number of pairs  $\times T(n)$
  - $n^2 \times \mathcal{O}(n^{n-1})$
  - $\mathcal{O}(n^2 n^{n-1})$

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**Algorithm 3** APSP-RECURSIVE-MEMOIZATION

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**Input:** A directed graph  $G = (V, E)$ **Output:** Shortest distance between all pair of vertices

```
1:  $n \leftarrow |G.V|$  ▷ Number of vertices in  $G$ 
2:  $m \leftarrow |G.E|$  ▷ Number of edges in  $G$ 
3: Create  $n$  distance matrices  $D^r[1, 2, \dots, n][1, 2, \dots, n]$  of size  $n \times n$  where  $0 \leq r \leq n - 1$ 
4: for  $r \leftarrow 0$  to  $n - 1$  do
5:   for each vertex  $u \in G.V$  do
6:     for each vertex  $v \in G.V$  do
7:        $D^r[u][v] \leftarrow -\infty$ 
8:     end for
9:   end for
10: end for
11: for each vertex  $u \in G.V$  do
12:   for each vertex  $v \in G.V$  do
13:      $D^{n-1}[u][v] \leftarrow \text{APSP-HELPER-MEMOIZATION}(n - 1, u, v)$  ▷ Weight of a shortest path from vertex  $u$  to vertex  $v$  that contains at most  $n - 1$  edges
14:   end for
15: end for
16: return  $D$ 
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**Algorithm 4** APSP-HELPER-MEMOIZATION( $r, u, v$ )

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```
1: if  $D^r[u][v] \neq -\infty$  then ▷ Weight of a shortest path from vertex  $u$  to vertex  $v$  that contains at most  $r$  edges is already computed
2:   return  $D^r[u][v]$ 
3: end if
4: if  $r = 0$  then
5:   if  $u = v$  then
6:     return 0 ▷ Weight of a shortest path from vertex  $u$  to vertex  $v$  that contains at most 0 edge is 0 because vertex  $u$  and vertex  $v$  are the same
7:   else
8:     return  $\infty$  ▷ Weight of a shortest path from vertex  $u$  to vertex  $v$  that contains at most 0 edge is  $\infty$  because vertex  $u$  and vertex  $v$  are different
9:   end if
10: else
```

Follow the formulation

$$D_{uv}^{(r)} = \min \left\{ \underbrace{D_{uk}^{(r-1)}}_{\text{Weight of a shortest path from } u \text{ to } k \text{ with atmost } r-1 \text{ edges}} + \underbrace{w(k, v)}_{\text{Weight of edge } (k, v)} : k \in G.V \right\} \quad (7)$$

```
11: minDistance  $\leftarrow \infty$ 
12: for each vertex  $k \in G.V$  do
13:    $\text{dist} \leftarrow \text{APSP-HELPER}(r - 1, u, k) + w(k, v)$ 
14:   if  $\text{dist} < \text{minDistance}$  then
15:      $\text{minDistance} \leftarrow \text{dist}$ 
16:   end if
17: end for
18:  $D^r[u][v] \leftarrow \text{minDistance}$ 
19: return  $\text{minDistance}$ 
20: end if
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**Algorithm 5** APSP-BOTTOM-UP

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**Input:** A directed graph  $G = (V, E)$ **Output:** Shortest distance between all pair of vertices

```
1:  $n \leftarrow |G.V|$  ▷ Number of vertices in  $G$ 
2:  $m \leftarrow |G.E|$  ▷ Number of edges in  $G$ 
3: Create  $n$  distance matrices  $D^r[1, 2, \dots, n][1, 2, \dots, n]$  of size  $n \times n$  where  $0 \leq r \leq n - 1$ 
4: for each vertex  $u \in G.V$  do
5:   for each vertex  $v \in G.V$  do
6:     if  $u = v$  then
7:        $D^0[u][v] \leftarrow 0$ 
8:     else
9:        $D^0[u][v] \leftarrow \infty$ 
10:    end if
11:  end for
12: end for
13: for  $r \leftarrow 1$  to  $n - 1$  do
14:   for each vertex  $u \in G.V$  do
15:    for each vertex  $v \in G.V$  do
16:       $\text{minDistance} \leftarrow \infty$ 
17:      for each vertex  $k \in G.V$  do
18:         $\text{dist} \leftarrow D^{r-1}[u][v] + w(k, v)$ 
19:        if  $\text{dist} < \text{minDistance}$  then
20:           $\text{minDistance} \leftarrow \text{dist}$ 
21:        end if
22:      end for
23:       $D^r[u][v] \leftarrow \text{minDistance}$ 
24:    end for
25:  end for
26: end for
27: return  $D^{n-1}$ 
```

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Obtain  $D^{(0)}$ :

$$D^{(0)} = \begin{bmatrix} 0 & \infty & \infty & \infty & \infty \\ \infty & 0 & \infty & \infty & \infty \\ \infty & \infty & 0 & \infty & \infty \\ \infty & \infty & \infty & 0 & \infty \\ \infty & \infty & \infty & \infty & 0 \end{bmatrix} \quad (8)$$

Obtain  $D^{(1)}$ :

$$D^{(1)} = \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix} \quad (9)$$

Obtain  $D^{(2)}$ :

$$D^{(2)} = \begin{bmatrix} 0 & 3 & 8 & \mathbf{2} & -4 \\ \mathbf{3} & 0 & \mathbf{-4} & 1 & 7 \\ \infty & 4 & 0 & \mathbf{5} & \mathbf{11} \\ 2 & \mathbf{-1} & -5 & 0 & \mathbf{-2} \\ \mathbf{8} & \infty & \mathbf{1} & 6 & 0 \end{bmatrix} \quad (10)$$

$D_{14}^{(2)} = \min \begin{cases} D_{11}^{(1)} + w(1, 4) = 0 + \infty = \infty \\ D_{12}^{(1)} + w(2, 4) = 3 + 1 = 4 \\ D_{13}^{(1)} + w(3, 4) = 8 + \infty = \infty \\ D_{14}^{(1)} + w(4, 4) = \infty + 0 = \infty \\ D_{15}^{(1)} + w(5, 4) = -4 + 6 = 2 \end{cases} = 2$	$D_{21}^{(2)} = \min \begin{cases} D_{21}^{(1)} + w(1, 1) = \infty + 0 = \infty \\ D_{22}^{(1)} + w(2, 1) = 0 + \infty = \infty \\ D_{23}^{(1)} + w(3, 1) = \infty + \infty = \infty \\ D_{24}^{(1)} + w(4, 1) = 1 + 2 = 3 \\ D_{25}^{(1)} + w(5, 1) = 7 + \infty = \infty \end{cases} = 3$
$D_{23}^{(2)} = \min \begin{cases} D_{21}^{(1)} + w(1, 3) = \infty + 8 = \infty \\ D_{22}^{(1)} + w(2, 3) = 0 + \infty = \infty \\ D_{23}^{(1)} + w(3, 3) = \infty + 0 = \infty \\ D_{24}^{(1)} + w(4, 3) = 1 - 5 = -4 \\ D_{25}^{(1)} + w(5, 3) = 7 + \infty = \infty \end{cases} = -4$	$D_{34}^{(2)} = \min \begin{cases} D_{31}^{(1)} + w(1, 4) = \infty + \infty = \infty \\ D_{32}^{(1)} + w(2, 4) = 4 + 1 = 5 \\ D_{33}^{(1)} + w(3, 4) = 0 + \infty = \infty \\ D_{34}^{(1)} + w(4, 4) = \infty + 0 = \infty \\ D_{35}^{(1)} + w(5, 4) = \infty + 6 = \infty \end{cases} = 5$
$D_{35}^{(2)} = \min \begin{cases} D_{31}^{(1)} + w(1, 5) = \infty - 4 = \infty \\ D_{32}^{(1)} + w(2, 5) = 4 + 7 = 11 \\ D_{33}^{(1)} + w(3, 5) = 0 + \infty = \infty \\ D_{34}^{(1)} + w(4, 5) = \infty + \infty = \infty \\ D_{35}^{(1)} + w(5, 5) = \infty + 0 = \infty \end{cases} = 11$	$D_{42}^{(2)} = \min \begin{cases} D_{41}^{(1)} + w(1, 2) = 2 + 3 = 5 \\ D_{42}^{(1)} + w(2, 2) = \infty + 0 = \infty \\ D_{43}^{(1)} + w(3, 2) = -5 + 4 = -1 \\ D_{44}^{(1)} + w(4, 2) = 0 + \infty = \infty \\ D_{45}^{(1)} + w(5, 2) = \infty + \infty = \infty \end{cases} = -1$
$D_{45}^{(2)} = \min \begin{cases} D_{41}^{(1)} + w(1, 5) = 2 - 4 = -2 \\ D_{42}^{(1)} + w(2, 5) = \infty + 7 = \infty \\ D_{43}^{(1)} + w(3, 5) = -5 + \infty = \infty \\ D_{44}^{(1)} + w(4, 5) = 0 + \infty = \infty \\ D_{45}^{(1)} + w(5, 5) = \infty + 0 = \infty \end{cases} = -2$	$D_{51}^{(2)} = \min \begin{cases} D_{51}^{(1)} + w(1, 1) = \infty + 0 = \infty \\ D_{52}^{(1)} + w(2, 1) = \infty + \infty = \infty \\ D_{53}^{(1)} + w(3, 1) = \infty + \infty = \infty \\ D_{54}^{(1)} + w(4, 1) = 6 + 2 = 8 \\ D_{55}^{(1)} + w(5, 1) = 0 + \infty = \infty \end{cases} = 8$
$D_{53}^{(2)} = \min \begin{cases} D_{51}^{(1)} + w(1, 3) = \infty + 8 = \infty \\ D_{52}^{(1)} + w(2, 3) = \infty + \infty = \infty \\ D_{53}^{(1)} + w(3, 3) = \infty + 0 = \infty \\ D_{54}^{(1)} + w(4, 3) = 6 - 5 = 1 \\ D_{55}^{(1)} + w(5, 3) = 0 + \infty = \infty \end{cases} = 1$	

Obtain  $D^{(3)}$ :

$$D^{(3)} = \begin{bmatrix} 0 & 3 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{bmatrix} \quad (11)$$

$D_{13}^{(3)} = \min \begin{cases} D_{11}^{(2)} + w(1, 3) = 0 + 8 & = 8 \\ D_{12}^{(2)} + w(2, 3) = 3 + \infty & = \infty \\ D_{13}^{(2)} + w(3, 3) = 8 + 0 & = 8 \\ D_{14}^{(2)} + w(4, 3) = 2 - 5 & = -3 \\ D_{15}^{(2)} + w(5, 3) = -4 + \infty & = \infty \end{cases} = -3$	$D_{25}^{(3)} = \min \begin{cases} D_{21}^{(2)} + w(1, 5) = 3 - 4 & = -1 \\ D_{22}^{(2)} + w(2, 5) = 0 + 7 & = 7 \\ D_{23}^{(2)} + w(3, 5) = -4 + \infty & = \infty \\ D_{24}^{(2)} + w(4, 5) = 1 + \infty & = \infty \\ D_{25}^{(2)} + w(5, 5) = 7 + 0 & = 7 \end{cases} = -1$
$D_{31}^{(3)} = \min \begin{cases} D_{31}^{(2)} + w(1, 1) = \infty + 0 & = \infty \\ D_{32}^{(2)} + w(2, 1) = 4 + \infty & = \infty \\ D_{33}^{(2)} + w(3, 1) = 0 + \infty & = \infty \\ D_{34}^{(2)} + w(4, 1) = 5 + 2 & = 7 \\ D_{35}^{(2)} + w(5, 1) = 11 + \infty & = \infty \end{cases} = 7$	$D_{52}^{(3)} = \min \begin{cases} D_{51}^{(2)} + w(1, 2) = 8 + 3 & = 11 \\ D_{52}^{(2)} + w(2, 2) = \infty + 0 & = \infty \\ D_{53}^{(2)} + w(3, 2) = 1 + 4 & = 5 \\ D_{54}^{(2)} + w(4, 2) = 6 + \infty & = \infty \\ D_{55}^{(2)} + w(5, 2) = 0 + \infty & = \infty \end{cases} = 5$

Obtain  $D^{(4)}$ :

$$D^{(4)} = \begin{bmatrix} 0 & 1 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{bmatrix} \quad (12)$$

$D_{12}^{(4)} = \min \begin{cases} D_{11}^{(3)} + w(1, 2) = 0 + 3 & = 3 \\ D_{12}^{(3)} + w(2, 2) = 3 + 0 & = 3 \\ D_{13}^{(3)} + w(3, 2) = -3 + 4 & = 1 \\ D_{14}^{(3)} + w(4, 2) = 2 + \infty & = \infty \\ D_{15}^{(3)} + w(5, 2) = -4 + \infty & = \infty \end{cases} = 1$	$D_{35}^{(4)} = \min \begin{cases} D_{31}^{(3)} + w(1, 5) = 0 + 3 & = 3 \\ D_{32}^{(3)} + w(2, 5) = 3 + 0 & = 3 \\ D_{33}^{(3)} + w(3, 5) = -3 + 4 & = 1 \\ D_{34}^{(3)} + w(4, 5) = 2 + \infty & = \infty \\ D_{35}^{(3)} + w(5, 5) = -4 + \infty & = \infty \end{cases} = 3$
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