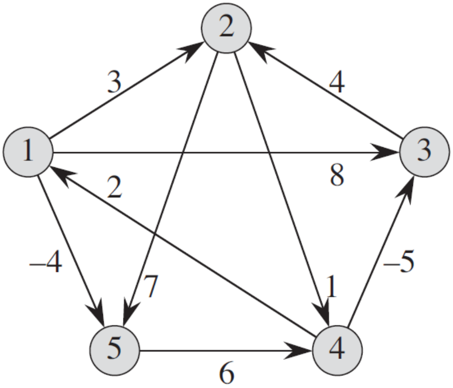
All Pair Shortest Path [From Cormen's Book]

April 20, 2025

All Pair Shortest Path

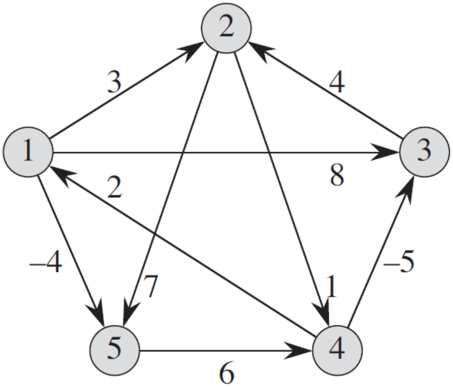


$D_{uv}^{(r)}$: Weight of a shortest path from vertex u to vertex v that contains at most r edges.

$$D_{uv}^{(0)} = \begin{cases} 0 & \text{if } u = v \\ \infty & \text{if } u \neq v \end{cases}$$

$$D^{(0)} = \begin{bmatrix} 0 & \infty & \infty & \infty & \infty \\ \infty & 0 & \infty & \infty & \infty \\ \infty & \infty & 0 & \infty & \infty \\ \infty & \infty & \infty & 0 & \infty \\ \infty & \infty & \infty & \infty & 0 \end{bmatrix}$$

All Pair Shortest Path



$D_{uv}^{(r)}$: Weight of a shortest path from vertex u to vertex v that contains at most r edges.

$$D_{uv}^{(1)} = \begin{cases} 0 & \text{if } u = v \\ w(u,v) & \text{if } (u,v) \in G.E \\ \infty & \text{otherwise} \end{cases}$$

$$D^{(1)} = \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$

All Pair Shortest Path

For $r \geq 1$,

$$D_{uv}^{(r)} = \min\left\{ \underbrace{D_{uv}^{(r-1)}}_{\substack{\text{Weight of a shortest path from} \\ u \text{ to } v \text{ with atmost } r-1 \text{ edges}}}, \min\left\{ \underbrace{D_{uk}^{(r-1)}}_{\substack{\text{Weight of a shortest path from} \\ u \text{ to } k \text{ with atmost } r-1 \text{ edges}}} + \underbrace{w(k,v)}_{\text{Weight of edge } (k,v)} : k \in \text{All predecessors of } v \right\}\right\} \quad (1)$$

- k belongs to the set of vertices for which there is an edge going to v.
- In a complete graph, $k \in G.V$.
- Considering $k \in G.V$ makes the formulation simple.
- It will not make any change to the shortest path because if $(k,v) \notin G.E$, then $w(k,v) = \infty$ and this ∞ will not affect the minimum calculation.

$$D_{uv}^{(r)} = \min\left\{ \underbrace{D_{uv}^{(r-1)}}_{\substack{\text{Weight of a shortest path from} \\ u \text{ to } v \text{ with atmost } r-1 \text{ edges}}}, \min\left\{ \underbrace{D_{uk}^{(r-1)}}_{\substack{\text{Weight of a shortest path from} \\ u \text{ to } k \text{ with atmost } r-1 \text{ edges}}} + \underbrace{w(k,v)}_{\text{Weight of edge } (k,v)} : k \in G.V \right\}\right\} \quad (2)$$

$$= \min\left\{ \underbrace{D_{uk}^{(r-1)}}_{\substack{\text{Weight of a shortest path from} \\ u \text{ to } k \text{ with atmost } r-1 \text{ edges}}} + \underbrace{w(k,v)}_{\text{Weight of edge } (k,v)} : k \in G.V \right\} \quad (3)$$

All Pair Shortest Path: Recursive Algorithm

Algorithm 1 APSP-RECURSIVE

Input: A directed graph $G = (V, E)$

Output: Shortest distance between all pair of vertices

```
1: n ← |G.V|                                   ▷ Number of vertices in G
2: m ← |G.E|                                   ▷ Number of edges in G
3: Create a distance matrix D[1,2,...,n][1,2,...,n] of size n × n
4: for each vertex u ∈ G.V do
5:     for each vertex v ∈ G.V do
6:         D[u][v] ← APSP-HELPER(n − 1, u, v)  ▷ Weight of a shortest path from vertex u
           to vertex v that contains at most n − 1 edges
7: return D
```

All Pair Shortest Path: Recursive Algorithm

Algorithm 2 APSP-HELPER(r, u, v)

```
if r = 0 then
    if u = v then
        return 0        ▷ Weight of a shortest path from vertex u to vertex v that contains
                          at most 0 edge is 0 because vertex u and vertex v are the same
    else
        return ∞        ▷ Weight of a shortest path from vertex u to vertex v that contains
                          at most 0 edge is ∞ because vertex u and vertex v are different
else
    minDistance ← ∞
    for each vertex k ∈ G.V do
        dist ← APSP-HELPER(r − 1, u, k) + w(k, v)
        if dist < minDistance then
            minDistance ← dist
    return minDistance
```

All Pair Shortest Path: Complexity Analysis

Recurrence Relation to Obtain the Shortest Path From Vertex u to Vertex v: Let $N = n$

$$T(n-1) = \underbrace{N}_{\substack{\text{No. of times APSP-HELPER} \\ \text{is called in line 9}}} \left[\underbrace{T(n-2)}_{\text{Time by APSP-HELPER in line 9}} + \underbrace{\mathcal{O}(1)}_{\text{Weight addition in line 9}} \right] + \mathcal{O}(1)$$

$$= NT(n-2) + \mathcal{O}(N)$$

$$= NT(n-2) + N$$

$$= N\left[NT(n-3) + N\right] + N$$

$$= N^2 T(n-3) + N^2 + N$$

$$= N^2\left[NT(n-4) + N\right] + N^2 + N$$

$$= N^3 T(n-4) + N^3 + N^2 + N$$

$$= N^3\left[NT(n-5) + N\right] + N^3 + N^2 + N$$

$$= N^4 T(n-5) + N^4 + N^3 + N^2 + N$$

$$= \vdots$$

All Pair Shortest Path: Complexity Analysis

$$
\begin{aligned}
T(n) &= N^{N-1}T(n-n) + N^{N-1} + N^{N-2} + \cdots + N^4 + N^3 + N^2 + N \\
&= N^{N-1}T(0) + N^{N-1} + N^{N-2} + \cdots + N^4 + N^3 + N^2 + N \\
&= N^{N-1} + N^{N-1} + N^{N-2} + \cdots + N^4 + N^3 + N^2 + N \\
&= N^{N-1} + N\frac{N^{N-1}-1}{N-1} \\
&= \mathcal{O}(N^{N-1}) \\
&= \mathcal{O}(n^{n-1}) \qquad (4)
\end{aligned}
$$

All Pair Shortest Path: Complexity Analysis

- Time complexity to obtain the shortest path from vertex u to vertex v
  - $T(n) = \mathcal{O}(n^{n-1})$
- Need to find the shortest path between each pair of vertices
- Time complexity to obtain the shortest path between each pair of vertices
  - Number of pairs $\times T(n)$
  - $n^2 \times \mathcal{O}(n^{n-1})$
  - $\mathcal{O}(n^2 n^{n-1})$

All Pair Shortest Path: Recursive Algorithm with Memoization

Algorithm 3 APSP-RECURSIVE-MEMOIZATION

Input: A directed graph $G = (V, E)$

Output: Shortest distance between all pair of vertices

```
n ← |G.V|                                          ▷ Number of vertices in G
m ← |G.E|                                          ▷ Number of edges in G
Create n distance matrices D^r[1, 2, ..., n][1, 2, ..., n] of size n × n where 0 ≤ r ≤ n − 1
for r ← 0 to n − 1 do
    for each vertex u ∈ G.V do
        for each vertex v ∈ G.V do
            D^r[u][v] ← −∞
for each vertex u ∈ G.V do
    for each vertex v ∈ G.V do
        D^{n−1}[u][v] ← APSP-HELPER-MEMOIZATION(n − 1, u, v)   ▷ Weight of a shortest
            path from vertex u to vertex v that contains at most n − 1 edges
return D
```

All Pair Shortest Path: Recursive Algorithm with Memoization

Algorithm 4 APSP-HELPER-MEMOIZATION(r, u, v)

if $D^r[u][v] \neq -\infty$ **then** ▷ Weight of a shortest path from vertex u to vertex v that contains at most r edges is already computed
    return $D^r[u][v]$
if $r = 0$ **then**
    if $u = v$ **then**
        $D^r[u][v] \leftarrow 0$
        return 0
    else
        $D^r[u][v] \leftarrow \infty$
        return ∞
else

All Pair Shortest Path: Recursive Algorithm with Memoization

```
    minDistance ← ∞
    for each vertex k ∈ G.V do
        dist ← APSP-HELPER(r − 1, u, k) + w(k, v)
        if dist < minDistance then
            minDistance ← dist
    D^r[u][v] ← minDistance
    return minDistance
```

Algorithm 5 APSP-BOTTOM-UP

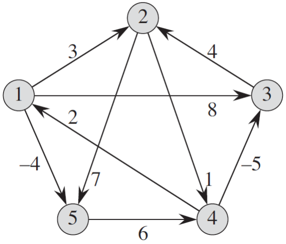
Input: A directed graph $G = (V, E)$
Output: Shortest distance between all pair of vertices

```
n ← |G.V|                                   ▷ Number of vertices in G
m ← |G.E|                                   ▷ Number of edges in G
Create n distance matrices D^r[1, 2, ..., n][1, 2, ..., n] of size n × n where 0 ≤ r ≤ n − 1
for each vertex u ∈ G.V do
    for each vertex v ∈ G.V do
        if u = v then
            D^0[u][v] ← 0
        else
            D^0[u][v] ← ∞
```

All Pair Shortest Path: Bottom-Up Algorithm

for $r \leftarrow 1$ **to** $n-1$ **do**
    for each vertex $u \in G.V$ **do**
        for each vertex $v \in G.V$ **do**
            $\texttt{minDistance} \leftarrow \infty$
            for each vertex $k \in G.V$ **do**
                $\texttt{dist} \leftarrow D^{r-1}[u][v] + w(k, v)$
                if $\texttt{dist} < \texttt{minDistance}$ **then**
                    $\texttt{minDistance} \leftarrow \texttt{dist}$
            $D^{r}[u][v] \leftarrow \texttt{minDistance}$
return D^{n-1}

Working Example

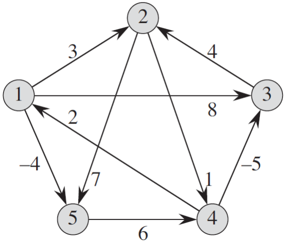


$$D^{(1)} = \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$

$$D^{(2)} = \begin{bmatrix} 0 & 3 & 8 & \mathbf{2} & -4 \\ \mathbf{3} & 0 & \mathbf{-4} & 1 & 7 \\ \infty & 4 & 0 & \mathbf{5} & \mathbf{11} \\ 2 & \mathbf{-1} & -5 & 0 & \mathbf{-2} \\ \mathbf{8} & \infty & \mathbf{1} & 6 & 0 \end{bmatrix}$$

$$D_{14}^{(2)} = \min \begin{cases} D_{11}^{(1)} + w(1,4) = 0 + \infty & = \infty \\ D_{12}^{(1)} + w(2,4) = 3 + 1 & = 4 \\ D_{13}^{(1)} + w(3,4) = 8 + \infty & = \infty \\ D_{14}^{(1)} + w(4,4) = \infty + 0 & = \infty \\ D_{15}^{(1)} + w(5,4) = -4 + 6 & = 2 \end{cases} = 2$$

Working Example

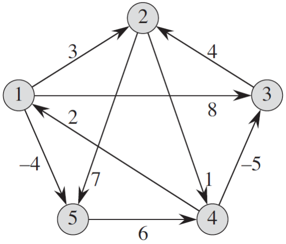


$$D^{(1)} = \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$

$$D^{(2)} = \begin{bmatrix} 0 & 3 & 8 & \mathbf{2} & -4 \\ \mathbf{3} & 0 & \mathbf{-4} & 1 & 7 \\ \infty & 4 & 0 & \mathbf{5} & \mathbf{11} \\ 2 & \mathbf{-1} & -5 & 0 & \mathbf{-2} \\ \mathbf{8} & \infty & \mathbf{1} & 6 & 0 \end{bmatrix}$$

$$D^{(2)}_{21} = \min \begin{cases} D^{(1)}_{21} + w(1,1) = \infty + 0 & = \infty \\ D^{(1)}_{22} + w(2,1) = 0 + \infty & = \infty \\ D^{(1)}_{23} + w(3,1) = \infty + \infty & = \infty \\ D^{(1)}_{24} + w(4,1) = 1 + 2 & = 3 \\ D^{(1)}_{25} + w(5,1) = 7 + \infty & = \infty \end{cases} = 3$$

Working Example

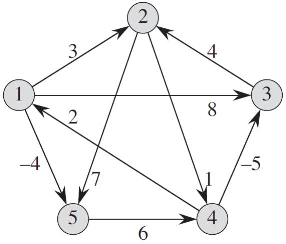


$$D^{(1)} = \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$

$$D^{(2)} = \begin{bmatrix} 0 & 3 & 8 & \mathbf{2} & -4 \\ \mathbf{3} & 0 & \mathbf{-4} & 1 & 7 \\ \infty & 4 & 0 & \mathbf{5} & \mathbf{11} \\ 2 & \mathbf{-1} & -5 & 0 & \mathbf{-2} \\ \mathbf{8} & \infty & \mathbf{1} & 6 & 0 \end{bmatrix}$$

$$D_{23}^{(2)} = \min \begin{cases} D_{21}^{(1)} + w(1,3) = \infty + 8 & = \infty \\ D_{22}^{(1)} + w(2,3) = 0 + \infty & = \infty \\ D_{23}^{(1)} + w(3,3) = \infty + 0 & = \infty \\ D_{24}^{(1)} + w(4,3) = 1 - 5 & = -4 \\ D_{25}^{(1)} + w(5,3) = 7 + \infty & = \infty \end{cases} = -4$$

Working Example

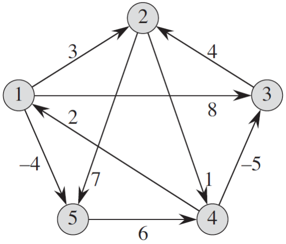


$$D^{(1)} = \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$

$$D^{(2)} = \begin{bmatrix} 0 & 3 & 8 & \mathbf{2} & -4 \\ \mathbf{3} & 0 & \mathbf{-4} & 1 & 7 \\ \infty & 4 & 0 & \mathbf{5} & \mathbf{11} \\ 2 & \mathbf{-1} & -5 & 0 & \mathbf{-2} \\ \mathbf{8} & \infty & \mathbf{1} & 6 & 0 \end{bmatrix}$$

$$D_{34}^{(2)} = \min \begin{cases} D_{31}^{(1)} + w(1,4) = \infty + \infty & = \infty \\ D_{32}^{(1)} + w(2,4) = 4 + 1 & = 5 \\ D_{33}^{(1)} + w(3,4) = 0 + \infty & = \infty \\ D_{34}^{(1)} + w(4,4) = \infty + 0 & = \infty \\ D_{35}^{(1)} + w(5,4) = \infty + 6 & = \infty \end{cases} = 5$$

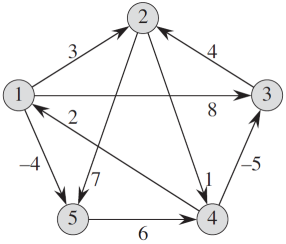
Working Example



$$D^{(2)} = \begin{bmatrix} 0 & 3 & 8 & \mathbf{2} & -4 \\ \mathbf{3} & 0 & \mathbf{-4} & 1 & 7 \\ \infty & 4 & 0 & \mathbf{5} & \mathbf{11} \\ 2 & \mathbf{-1} & -5 & 0 & \mathbf{-2} \\ \mathbf{8} & \infty & \mathbf{1} & 6 & 0 \end{bmatrix}$$

$$D^{(1)} = \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$

$$D^{(2)}_{35} = \min \begin{cases} D^{(1)}_{31} + w(1,5) = \infty - 4 & = \infty \\ D^{(1)}_{32} + w(2,5) = 4 + 7 & = 11 \\ D^{(1)}_{33} + w(3,5) = 0 + \infty & = \infty \\ D^{(1)}_{34} + w(4,5) = \infty + \infty & = \infty \\ D^{(1)}_{35} + w(5,5) = \infty + 0 & = \infty \end{cases} = 11$$

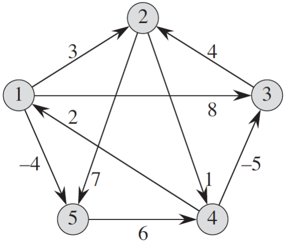


$$D^{(1)} = \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$

$$D^{(2)} = \begin{bmatrix} 0 & 3 & 8 & \mathbf{2} & -4 \\ \mathbf{3} & 0 & \mathbf{-4} & 1 & 7 \\ \infty & 4 & 0 & \mathbf{5} & \mathbf{11} \\ 2 & \mathbf{-1} & -5 & 0 & \mathbf{-2} \\ \mathbf{8} & \infty & \mathbf{1} & 6 & 0 \end{bmatrix}$$

$$D_{42}^{(2)} = \min \begin{cases} D_{41}^{(1)} + w(1,2) = 2 + 3 & = 5 \\ D_{42}^{(1)} + w(2,2) = \infty + 0 & = \infty \\ D_{43}^{(1)} + w(3,2) = -5 + 4 & = -1 \\ D_{44}^{(1)} + w(4,2) = 0 + \infty & = \infty \\ D_{45}^{(1)} + w(5,2) = \infty + \infty & = \infty \end{cases} = -1$$

Working Example

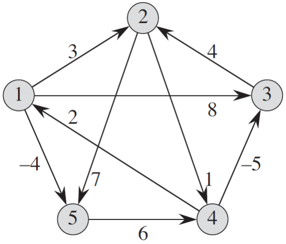


$$D^{(2)} = \begin{bmatrix} 0 & 3 & 8 & \mathbf{2} & -4 \\ \mathbf{3} & 0 & \mathbf{-4} & 1 & 7 \\ \infty & 4 & 0 & \mathbf{5} & \mathbf{11} \\ 2 & \mathbf{-1} & -5 & 0 & \mathbf{-2} \\ \mathbf{8} & \infty & \mathbf{1} & 6 & 0 \end{bmatrix}$$

$$D^{(1)} = \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$

$$D^{(2)}_{45} = \min \begin{cases} D^{(1)}_{41} + w(1,5) = 2 - 4 & = -2 \\ D^{(1)}_{42} + w(2,5) = \infty + 7 & = \infty \\ D^{(1)}_{43} + w(3,5) = -5 + \infty & = \infty \\ D^{(1)}_{44} + w(4,5) = 0 + \infty & = \infty \\ D^{(1)}_{45} + w(5,5) = \infty + 0 & = \infty \end{cases} = -2$$

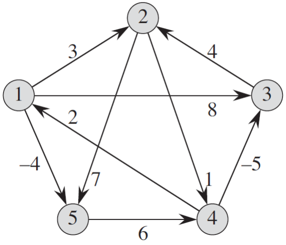
Working Example



$$D^{(1)} = \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$

$$D^{(2)} = \begin{bmatrix} 0 & 3 & 8 & \mathbf{2} & -4 \\ \mathbf{3} & 0 & \mathbf{-4} & 1 & 7 \\ \infty & 4 & 0 & \mathbf{5} & \mathbf{11} \\ 2 & \mathbf{-1} & -5 & 0 & \mathbf{-2} \\ \mathbf{8} & \infty & \mathbf{1} & 6 & 0 \end{bmatrix}$$

$$D^{(2)}_{51} = \min \begin{cases} D^{(1)}_{51} + w(1,1) = \infty + 0 & = \infty \\ D^{(1)}_{52} + w(2,1) = \infty + \infty & = \infty \\ D^{(1)}_{53} + w(3,1) = \infty + \infty & = \infty \\ D^{(1)}_{54} + w(4,1) = 6 + 2 & = 8 \\ D^{(1)}_{55} + w(5,1) = 0 + \infty & = \infty \end{cases} = 8$$

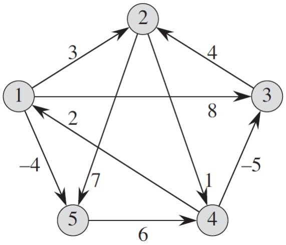


$$D^{(2)} = \begin{bmatrix} 0 & 3 & 8 & \mathbf{2} & -4 \\ \mathbf{3} & 0 & \mathbf{-4} & 1 & 7 \\ \infty & 4 & 0 & \mathbf{5} & \mathbf{11} \\ 2 & \mathbf{-1} & -5 & 0 & \mathbf{-2} \\ \mathbf{8} & \infty & \mathbf{1} & 6 & 0 \end{bmatrix}$$

$$D^{(1)} = \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$

$$D^{(2)}_{53} = \min \begin{cases} D^{(1)}_{51} + w(1,3) = \infty + 8 & = \infty \\ D^{(1)}_{52} + w(2,3) = \infty + \infty & = \infty \\ D^{(1)}_{53} + w(3,3) = \infty + 0 & = \infty \\ D^{(1)}_{54} + w(4,3) = 6 - 5 & = 1 \\ D^{(1)}_{55} + w(5,3) = 0 + \infty & = \infty \end{cases} = 1$$

Working Example

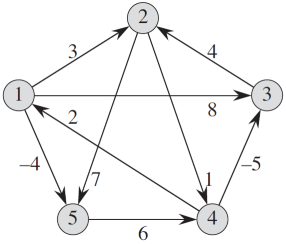


$$D^{(2)} = \begin{bmatrix} 0 & 3 & 8 & 2 & -4 \\ 3 & 0 & -4 & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & \infty & 1 & 6 & 0 \end{bmatrix}$$

$$D^{(3)} = \begin{bmatrix} 0 & 3 & \mathbf{-3} & 2 & -4 \\ 3 & 0 & -4 & 1 & \mathbf{-1} \\ \mathbf{7} & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & \mathbf{5} & 1 & 6 & 0 \end{bmatrix}$$

$$D^{(3)}_{13} = \min \begin{cases} D^{(2)}_{11} + w(1,3) = 0 + 8 & = 8 \\ D^{(2)}_{12} + w(2,3) = 3 + \infty & = \infty \\ D^{(2)}_{13} + w(3,3) = 8 + 0 & = 8 \\ D^{(2)}_{14} + w(4,3) = 2 - 5 & = -3 \\ D^{(2)}_{15} + w(5,3) = -4 + \infty & = \infty \end{cases} = -3$$

Working Example

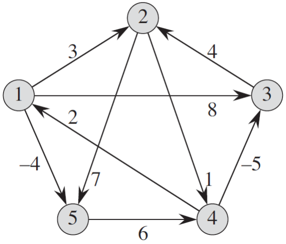


$$D^{(3)} = \begin{bmatrix} 0 & 3 & \mathbf{-3} & 2 & -4 \\ 3 & 0 & -4 & 1 & \mathbf{-1} \\ \mathbf{7} & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & \mathbf{5} & 1 & 6 & 0 \end{bmatrix}$$

$$D^{(2)} = \begin{bmatrix} 0 & 3 & 8 & 2 & -4 \\ 3 & 0 & -4 & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & \infty & 1 & 6 & 0 \end{bmatrix}$$

$$D_{25}^{(3)} = \min \begin{cases} D_{21}^{(2)} + w(1,5) = 3 - 4 & = -1 \\ D_{22}^{(2)} + w(2,5) = 0 + 7 & = 7 \\ D_{23}^{(2)} + w(3,5) = -4 + \infty & = \infty \\ D_{24}^{(2)} + w(4,5) = 1 + \infty & = \infty \\ D_{25}^{(2)} + w(5,5) = 7 + 0 & = 7 \end{cases} = -1$$

Working Example

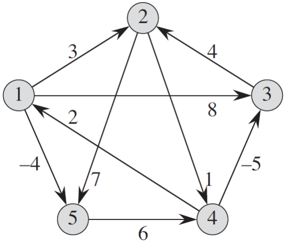


$$D^{(3)} = \begin{bmatrix} 0 & 3 & \mathbf{-3} & 2 & -4 \\ 3 & 0 & -4 & 1 & \mathbf{-1} \\ \mathbf{7} & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & \mathbf{5} & 1 & 6 & 0 \end{bmatrix}$$

$$D^{(2)} = \begin{bmatrix} 0 & 3 & 8 & 2 & -4 \\ 3 & 0 & -4 & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & \infty & 1 & 6 & 0 \end{bmatrix}$$

$$D_{31}^{(3)} = \min \begin{cases} D_{31}^{(2)} + w(1,1) = \infty + 0 & = \infty \\ D_{32}^{(2)} + w(2,1) = 4 + \infty & = \infty \\ D_{33}^{(2)} + w(3,1) = 0 + \infty & = \infty \\ D_{34}^{(2)} + w(4,1) = 5 + 2 & = 7 \\ D_{35}^{(2)} + w(5,1) = 11 + \infty & = \infty \end{cases} = 7$$

Working Example

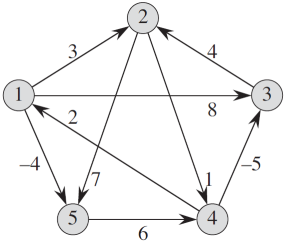


$$D^{(2)} = \begin{bmatrix} 0 & 3 & 8 & 2 & -4 \\ 3 & 0 & -4 & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & \infty & 1 & 6 & 0 \end{bmatrix}$$

$$D^{(3)} = \begin{bmatrix} 0 & 3 & \mathbf{-3} & 2 & -4 \\ 3 & 0 & -4 & 1 & \mathbf{-1} \\ \mathbf{7} & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & \mathbf{5} & 1 & 6 & 0 \end{bmatrix}$$

$$D^{(3)}_{52} = \min \begin{cases} D^{(2)}_{51} + w(1,2) = 8 + 3 & = 11 \\ D^{(2)}_{52} + w(2,2) = \infty + 0 & = \infty \\ D^{(2)}_{53} + w(3,2) = 1 + 4 & = 5 \\ D^{(2)}_{54} + w(4,2) = 6 + \infty & = \infty \\ D^{(2)}_{55} + w(5,2) = 0 + \infty & = \infty \end{cases} = 5$$

Working Example

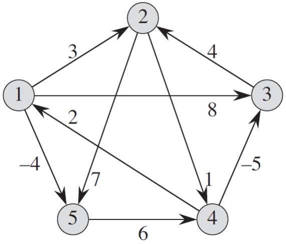


$$D^{(4)} = \begin{bmatrix} 0 & 1 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{bmatrix}$$

$$D^{(3)} = \begin{bmatrix} 0 & 3 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{bmatrix}$$

$$D^{(4)}_{12} = \min \begin{cases} D^{(3)}_{11} + w(1,2) = 0 + 3 & = 3 \\ D^{(3)}_{12} + w(2,2) = 3 + 0 & = 3 \\ D^{(3)}_{13} + w(3,2) = -3 + 4 & = 1 \\ D^{(3)}_{14} + w(4,2) = 2 + \infty & = \infty \\ D^{(3)}_{15} + w(5,2) = -4 + \infty & = \infty \end{cases} = 1$$

Working Example

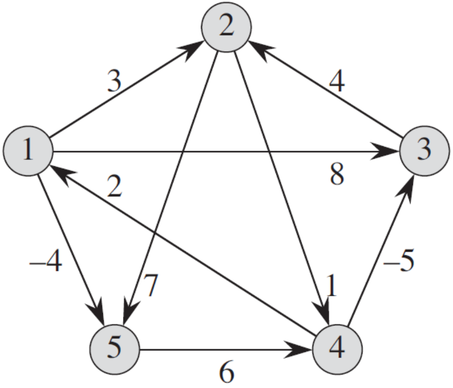


$$D^{(3)} = \begin{bmatrix} 0 & 3 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{bmatrix}$$

$$D^{(4)} = \begin{bmatrix} 0 & \mathbf{1} & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & \mathbf{3} \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{bmatrix}$$

$$D^{(4)}_{35} = \min \begin{cases} D^{(3)}_{31} + w(1,5) = 0 + 3 & = 3 \\ D^{(3)}_{32} + w(2,5) = 3 + 0 & = 3 \\ D^{(3)}_{33} + w(3,5) = -3 + 4 & = 1 \\ D^{(3)}_{34} + w(4,5) = 2 + \infty & = \infty \\ D^{(3)}_{35} + w(5,5) = -4 + \infty & = \infty \end{cases} = 3$$

Floyd-Warshall Algorithm



$D_{uv}^{(k)}$: Weight of a shortest path from vertex u to vertex v for which all intermediate vertices are in the set $\{1, 2, \ldots, k\}$.

$$D_{uv}^{(0)} = \begin{cases} 0 & \text{if } u = v \\ w(u,v) & \text{if } (u,v) \in G.E \\ \infty & \text{otherwise} \end{cases}$$

$$D^{(0)} = \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$

Floyd-Warshall Algorithm

For $k \geq 1$,

$$D_{uv}^{(k)} = \min \begin{cases} \underbrace{D_{uv}^{(k-1)}}_{\substack{\text{Weight of a shortest path from } u \text{ to } v \text{ for which} \\ \text{all intermediate vertices are in the set } \{1,2,\ldots,k-1\}}} \\ \underbrace{D_{uk}^{(k-1)}}_{\substack{\text{Weight of a shortest path from } u \text{ to } k \text{ for which} \\ \text{all intermediate vertices are in the set } \{1,2,\ldots,k-1\}}} + \underbrace{D_{kv}^{(k-1)}}_{\substack{\text{Weight of a shortest path from } k \text{ to } v \text{ for which} \\ \text{all intermediate vertices are in the set } \{1,2,\ldots,k-1\}}} \end{cases} \tag{5}$$

Floyd-Warshall Algorithm: Recursive

Algorithm 6 FLOYD-WARSHALL-RECURSIVE

Input: A directed graph $G = (V, E)$

Output: Shortest distance between all pair of vertices

```
1: n ← |G.V|                                    ▷ Number of vertices in G
2: m ← |G.E|                                    ▷ Number of edges in G
3: Create a distance matrix D[1, 2, ..., n][1, 2, ..., n] of size n × n
4: for each u ∈ G.V do
5:     for each v ∈ G.V do
6:         D[u][v] ← FW-HELPER(n, u, v)         ▷ Weight of a shortest path from vertex u
           to vertex v for which all intermediate vertices are in the set {1, 2, ..., n}
7: return D
```

Floyd-Warshall Algorithm: Recursive

Algorithm 7 FW-HELPER

```
if k = 0 then
    if u = v then
        return 0
    else if (u, v) ∈ G.E then
        return w(u, v)
    else
        return ∞
else
```

$$\textbf{return} \ \min \begin{cases} \text{FW-HELPER}(k-1, u, v) \\ \text{FW-HELPER}(k-1, u, k) + \text{FW-HELPER}(k-1, k, v) \end{cases}$$

Recurrence Relation to Obtain the Shortest Path From Vertex u to Vertex v:

$$
\begin{aligned}
T(n) &= \underbrace{3}_{\text{No. of times FW-HELPER is called in line 10}} \times \underbrace{T(n-1)}_{\text{Time by FW-HELPER in line 10}} + \underbrace{\mathcal{O}(1)}_{\text{Other factors}} \\
&= 3T(n-1) + 1 \\
&= 3\left[3T(n-2) + 1\right] + 1 \\
&= 3^2 T(n-2) + [3+1] \\
&= 3^2 \left[3T(n-3) + 1\right] + [3+1] \\
&= 3^3 T(n-3) + \left[3^2 + 3 + 1\right] \\
&= 3^3 \left[3T(n-4) + 1\right] + \left[3^2 + 3 + 1\right] \\
&= 3^4 T(n-4) + \left[3^3 + 3^2 + 3 + 1\right] \\
&\vdots
\end{aligned}
$$

Floyd-Warshall Algorithm: Complexity Analysis

$$\begin{aligned}
T(n) &= 3^n T(n-n) + \left[3^{n-1} + 2^{n-2} + \cdots + 3^3 + 3^2 + 3 + 1\right] \\
&= 3^n T(0) + \left[3^{n-1} + 2^{n-2} + \cdots + 3^3 + 3^2 + 3 + 1\right] \\
&= 3^n + \left[3^{n-1} + 2^{n-2} + \cdots + 3^3 + 3^2 + 3 + 1\right] \\
&= 3^n + 3^{n-1} + 2^{n-2} + \cdots + 3^3 + 3^2 + 3 + 1 \\
&= \frac{1}{2}(3^{n+1} - 1) \\
&= \mathcal{O}(3^{n+1})
\end{aligned} \tag{6}$$

Floyd-Warshall Algorithm: Complexity Analysis

- Time complexity to obtain the shortest path from vertex u to vertex v
  - $T(n) = \mathcal{O}(3^{n+1})$
- Need to find the shortest path between each pair of vertices
- Time complexity to obtain the shortest path between each pair of vertices
  - Number of pairs $\times T(n)$
  - $n^2 \times \mathcal{O}(3^{n+1})$
  - $\mathcal{O}(n^2 3^{n+1})$

Floyd-Warshall Algorithm: Recursive Algorithm with Memoization

Algorithm 8 FW-RECURSIVE-MEMOIZATION

Input: A directed graph $G = (V, E)$

Output: Shortest distance between all pair of vertices

```
n ← |G.V|                                          ▷ Number of vertices in G
m ← |G.E|                                          ▷ Number of edges in G
Create n + 1 distance matrices D^k[1, 2, ..., n][1, 2, ..., n] of size n × n where 0 ≤ k ≤ n
for k ← 0 to n do
    for each vertex u ∈ G.V do
        for each vertex v ∈ G.V do
            D^k[u][v] ← −∞
for each vertex u ∈ G.V do
    for each vertex v ∈ G.V do
        D^n[u][v] ← FW-HELPER-MEMOIZATION(n, u, v)    ▷ Weight of a shortest path
            from vertex u to vertex v for which all intermediate vertices are in the set {1, 2, ..., n}
return D
```

Floyd-Warshall Algorithm: Recursive Algorithm with Memoization

Algorithm 9 FW-HELPER-MEMOIZATION(k, u, v)

if $D^k[u][v] \neq -\infty$ **then** ▷ Weight of a shortest path from vertex u to vertex v for which all intermediate vertices are in the set $\{1, 2, \ldots, k\}$ is already computed

    return $D^k[u][v]$

if $k = 0$ **then**

    if $u = v$ **then**

        $D^k[u][v] \leftarrow 0$

        return 0

    else if $(u, v) \in G.E$ **then**

        $D^k[u][v] \leftarrow w(u, v)$

        return $w(u, v)$

    else

        $D^k[u][v] \leftarrow \infty$

        return ∞

else

Floyd-Warshall Algorithm: Recursive Algorithm with Memoization

$$D^k[u][v] = \min \begin{cases} \text{FW-HELPER}(k-1, u, v) \\ \text{FW-HELPER}(k-1, u, k) + \text{FW-HELPER}(k-1, k, v) \end{cases}$$

return $D^k[u][v]$

Algorithm 10 APSP-BOTTOM-UP

Input: A directed graph $G = (V, E)$

Output: Shortest distance between all pair of vertices

```
n ← |G.V|                                    ▷ Number of vertices in G
m ← |G.E|                                    ▷ Number of edges in G
Create n distance matrices D^k[1,2,...,n][1,2,...,n] of size n × n where 0 ≤ k ≤ n
for each vertex u ∈ G.V do
    for each vertex v ∈ G.V do
        if u = v then
            D^0[u][v] ← 0
        else if (u, v) ∈ G.E then
            D^0[u][v] ← w(u, v)
        else
            D^0[u][v] ← ∞
```

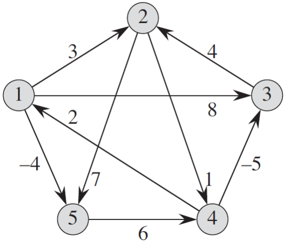
Floyd-Warshall Algorithm:: Bottom-Up Algorithm

for $k \leftarrow 1$ **to** n **do**
    for each vertex $u \in G.V$ **do**
        for each vertex $v \in G.V$ **do**

$$D^k[u][v] = \min \begin{cases} D^{k-1}[u][v \\ D^{k-1}[u][k] + D^{k-1}[k][v] \end{cases}$$

return D^n

Working Example



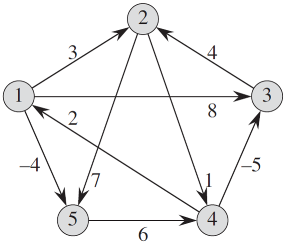
$$D^{(0)} = \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$

$$D^{(1)} = \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \mathbf{5} & -5 & 0 & \mathbf{-2} \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$

$$D^{(1)}_{42} = \min \begin{cases} D^{(0)}_{42} & = \infty \\ D^{(0)}_{41} + D^{(0)}_{12} & = 2 + 3 \end{cases} = 5$$

$$D^{(1)}_{45} = \min \begin{cases} D^{(0)}_{45} & = \infty \\ D^{(0)}_{41} + D^{(0)}_{15} & = 2 - 4 \end{cases} = -2$$

Working Example



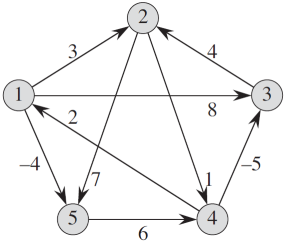
$$\Pi^{(0)} = \begin{bmatrix} \text{NIL} & 1 & 1 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\ \text{NIL} & 3 & \text{NIL} & \text{NIL} & \text{NIL} \\ 4 & \text{NIL} & 4 & \text{NIL} & \text{NIL} \\ \text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \end{bmatrix}$$

$$\Pi^{(1)} = \begin{bmatrix} \text{NIL} & 1 & 1 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\ \text{NIL} & 3 & \text{NIL} & \text{NIL} & \text{NIL} \\ 4 & \mathbf{1} & 4 & \text{NIL} & \mathbf{1} \\ \text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \end{bmatrix}$$

$$\Pi^{(1)}_{42} = \begin{cases} \Pi^{(0)}_{42} & \text{if } D^{(0)}_{42} \leq D^{(0)}_{41} + D^{(0)}_{12} \\ \Pi^{(0)}_{12} & \text{if } D^{(0)}_{42} > D^{(0)}_{41} + D^{(0)}_{12} \end{cases} = 1$$

$$\Pi^{(1)}_{45} = \begin{cases} \Pi^{(0)}_{45} & \text{if } D^{(0)}_{45} \leq D^{(0)}_{41} + D^{(0)}_{15} \\ \Pi^{(0)}_{15} & \text{if } D^{(0)}_{45} > D^{(0)}_{41} + D^{(0)}_{15} \end{cases} = 1$$

Working Example



$$D^{(1)} = \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$

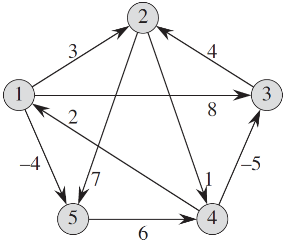
$$D^{(2)} = \begin{bmatrix} 0 & 3 & 8 & \mathbf{4} & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \mathbf{5} & \mathbf{11} \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$

$$D^{(2)}_{14} = \min \begin{cases} D^{(1)}_{14} & = \infty \\ D^{(1)}_{12} + D^{(1)}_{24} & = 3 + 1 \end{cases} = 4$$

$$D^{(2)}_{34} = \min \begin{cases} D^{(1)}_{34} & = \infty \\ D^{(1)}_{32} + D^{(1)}_{24} & = 4 + 1 \end{cases} = 5$$

$$D^{(2)}_{35} = \min \begin{cases} D^{(1)}_{35} & = \infty \\ D^{(1)}_{32} + D^{(1)}_{25} & = 4 + 7 \end{cases} = 11$$

Working Example



$$\Pi^{(1)} = \begin{bmatrix} \text{NIL} & 1 & 1 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\ \text{NIL} & 3 & \text{NIL} & \text{NIL} & \text{NIL} \\ 4 & 1 & 4 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \end{bmatrix}$$

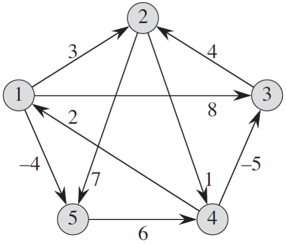
$$\Pi^{(2)} = \begin{bmatrix} \text{NIL} & 1 & 1 & \mathbf{2} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\ \text{NIL} & 3 & \text{NIL} & \mathbf{2} & \mathbf{2} \\ 4 & 1 & 4 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \end{bmatrix}$$

$$\Pi^{(2)}_{14} = \begin{cases} \Pi^{(1)}_{14} & \text{if } D^{(1)}_{42} \leq D^{(1)}_{12} + D^{(1)}_{24} \\ \Pi^{(1)}_{24} & \text{if } D^{(1)}_{42} > D^{(1)}_{12} + D^{(1)}_{24} \end{cases} = 2$$

$$\Pi^{(2)}_{34} = \begin{cases} \Pi^{(1)}_{34} & \text{if } D^{(1)}_{34} \leq D^{(1)}_{32} + D^{(1)}_{24} \\ \Pi^{(1)}_{24} & \text{if } D^{(1)}_{34} > D^{(1)}_{32} + D^{(1)}_{24} \end{cases} = 2$$

$$\Pi^{(2)}_{35} = \begin{cases} \Pi^{(1)}_{35} & \text{if } D^{(1)}_{35} \leq D^{(1)}_{32} + D^{(1)}_{25} \\ \Pi^{(1)}_{25} & \text{if } D^{(1)}_{35} > D^{(1)}_{32} + D^{(1)}_{25} \end{cases} = 2$$

Working Example

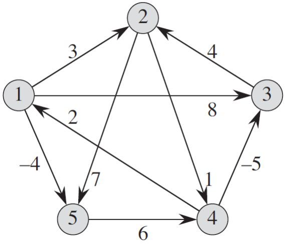


$$D^{(3)} = \begin{bmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & \mathbf{-1} & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$

$$D^{(2)} = \begin{bmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$

$$D^{(3)}_{42} = \min \begin{cases} D^{(2)}_{42} & = 5 \\ D^{(2)}_{43} + D^{(2)}_{32} & = -5 + 4 \end{cases} = -1$$

Working Example

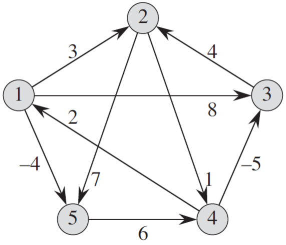


$$\Pi^{(3)} = \begin{bmatrix} \mathrm{NIL} & 1 & 1 & 2 & 1 \\ \mathrm{NIL} & \mathrm{NIL} & \mathrm{NIL} & 2 & 2 \\ \mathrm{NIL} & 3 & \mathrm{NIL} & 2 & 2 \\ 4 & \mathbf{3} & 4 & \mathrm{NIL} & 1 \\ \mathrm{NIL} & \mathrm{NIL} & \mathrm{NIL} & 5 & NIL \end{bmatrix}$$

$$\Pi^{(2)} = \begin{bmatrix} \mathrm{NIL} & 1 & 1 & 2 & 1 \\ \mathrm{NIL} & \mathrm{NIL} & \mathrm{NIL} & 2 & 2 \\ \mathrm{NIL} & 3 & \mathrm{NIL} & 2 & 2 \\ 4 & 1 & 4 & \mathrm{NIL} & 1 \\ \mathrm{NIL} & \mathrm{NIL} & \mathrm{NIL} & 5 & \mathrm{NIL} \end{bmatrix}$$

$$\Pi^{(3)}_{42} = \begin{cases} \Pi^{(2)}_{42} & \text{if } D^{(2)}_{42} \leq D^{(2)}_{43} + D^{(2)}_{32} \\ \Pi^{(2)}_{32} & \text{if } D^{(2)}_{42} > D^{(2)}_{43} + D^{(2)}_{32} \end{cases} = 3$$

Working Example



$$D^{(3)} = \begin{bmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$

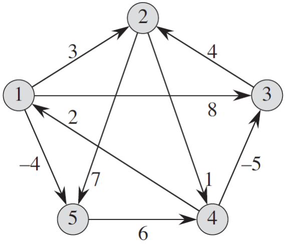
$$D^{(4)} = \begin{bmatrix} 0 & 3 & \mathbf{-1} & 4 & -4 \\ \mathbf{3} & 0 & \mathbf{-4} & 1 & \mathbf{-1} \\ \mathbf{7} & 4 & 0 & 5 & \mathbf{3} \\ 2 & -1 & -5 & 0 & -2 \\ \mathbf{8} & \mathbf{5} & \mathbf{1} & 6 & 0 \end{bmatrix}$$

$$D^{(4)}_{13} = \min \begin{cases} D^{(3)}_{13} & = 8 \\ D^{(3)}_{14} + D^{(3)}_{43} & = 4 - 5 \end{cases} = -1$$

$$D^{(4)}_{21} = \min \begin{cases} D^{(3)}_{21} & = \infty \\ D^{(3)}_{24} + D^{(3)}_{41} & = 1 + 2 \end{cases} = 3$$

$$D^{(4)}_{23} = \min \begin{cases} D^{(3)}_{23} & = \infty \\ D^{(3)}_{24} + D^{(3)}_{43} & = 1 - 5 \end{cases} = -4$$

Working Example



$$D^{(3)} = \begin{bmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$

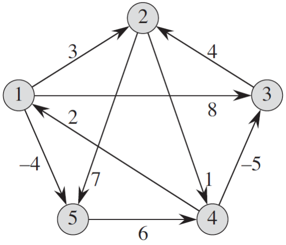
$$D^{(4)} = \begin{bmatrix} 0 & 3 & \mathbf{-1} & 4 & -4 \\ \mathbf{3} & 0 & \mathbf{-4} & 1 & \mathbf{-1} \\ \mathbf{7} & 4 & 0 & 5 & \mathbf{3} \\ 2 & -1 & -5 & 0 & -2 \\ \mathbf{8} & \mathbf{5} & \mathbf{1} & 6 & 0 \end{bmatrix}$$

$$D_{25}^{(4)} = \min \begin{cases} D_{25}^{(3)} & = 7 \\ D_{24}^{(3)} + D_{45}^{(3)} & = 1 - 2 \end{cases} = -1$$

$$D_{31}^{(4)} = \min \begin{cases} D_{31}^{(3)} & = \infty \\ D_{34}^{(3)} + D_{41}^{(3)} & = 5 + 2 \end{cases} = 7$$

$$D_{35}^{(4)} = \min \begin{cases} D_{35}^{(3)} & = 11 \\ D_{34}^{(3)} + D_{45}^{(3)} & = 5 - 2 \end{cases} = 3$$

Working Example



$$D^{(3)} = \begin{bmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$

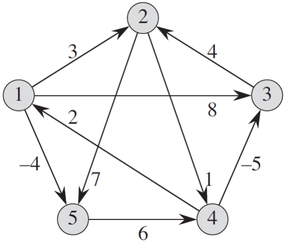
$$D^{(4)} = \begin{bmatrix} 0 & 3 & \mathbf{-1} & 4 & -4 \\ \mathbf{3} & 0 & \mathbf{-4} & 1 & \mathbf{-1} \\ \mathbf{7} & 4 & 0 & 5 & \mathbf{3} \\ 2 & -1 & -5 & 0 & -2 \\ \mathbf{8} & \mathbf{5} & \mathbf{1} & 6 & 0 \end{bmatrix}$$

$$D^{(4)}_{51} = \min \begin{cases} D^{(3)}_{51} & = \infty \\ D^{(3)}_{54} + D^{(3)}_{41} & = 6 + 2 \end{cases} = 8$$

$$D^{(4)}_{52} = \min \begin{cases} D^{(3)}_{52} & = \infty \\ D^{(3)}_{54} + D^{(3)}_{42} & = 6 - 1 \end{cases} = 5$$

$$D^{(4)}_{53} = \min \begin{cases} D^{(3)}_{53} & = \infty \\ D^{(3)}_{54} + D^{(3)}_{43} & = 6 - 5 \end{cases} = 1$$

Working Example



$$\Pi^{(3)} = \begin{bmatrix} NIL & 1 & 1 & 2 & 1 \\ NIL & NIL & NIL & 2 & 2 \\ NIL & 3 & NIL & 2 & 2 \\ 4 & 3 & 4 & NIL & 1 \\ NIL & NIL & NIL & 5 & NIL \end{bmatrix}$$

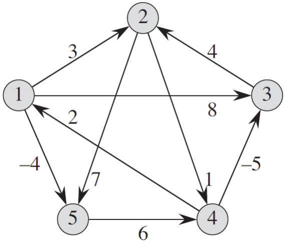
$$\Pi^{(4)} = \begin{bmatrix} NIL & 1 & 4 & 2 & 1 \\ 4 & NIL & 4 & 2 & 1 \\ 4 & 3 & NIL & 2 & 1 \\ 4 & 3 & 4 & NIL & 1 \\ 4 & 3 & 4 & 5 & NIL \end{bmatrix}$$

$$\Pi^{(4)}_{13} = \begin{cases} \Pi^{(3)}_{13} & \text{if } D^{(3)}_{13} \leq D^{(3)}_{14} + D^{(3)}_{43} \\ \Pi^{(3)}_{43} & \text{if } D^{(3)}_{13} > D^{(3)}_{14} + D^{(3)}_{43} \end{cases} = 4$$

$$\Pi^{(4)}_{21} = \begin{cases} \Pi^{(3)}_{21} & \text{if } D^{(3)}_{21} \leq D^{(3)}_{24} + D^{(3)}_{41} \\ \Pi^{(3)}_{41} & \text{if } D^{(3)}_{21} > D^{(3)}_{24} + D^{(3)}_{41} \end{cases} = 4$$

$$\Pi^{(4)}_{23} = \begin{cases} \Pi^{(3)}_{23} & \text{if } D^{(3)}_{23} \leq D^{(3)}_{24} + D^{(3)}_{43} \\ \Pi^{(3)}_{43} & \text{if } D^{(3)}_{23} > D^{(3)}_{24} + D^{(3)}_{43} \end{cases} = 4$$

Working Example



$$\Pi^{(3)} = \begin{bmatrix} NIL & 1 & 1 & 2 & 1 \\ NIL & NIL & NIL & 2 & 2 \\ NIL & 3 & NIL & 2 & 2 \\ 4 & 3 & 4 & NIL & 1 \\ NIL & NIL & NIL & 5 & NIL \end{bmatrix}$$

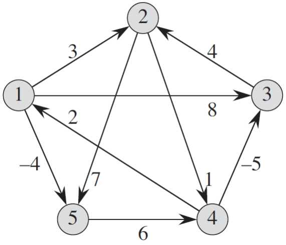
$$\Pi^{(4)} = \begin{bmatrix} NIL & 1 & \color{red}{4} & 2 & 1 \\ \color{red}{4} & NIL & \color{red}{4} & 2 & \color{red}{1} \\ \color{red}{4} & 3 & NIL & 2 & \color{red}{1} \\ 4 & 3 & 4 & NIL & 1 \\ \color{red}{4} & \color{red}{3} & \color{red}{4} & 5 & NIL \end{bmatrix}$$

$$\Pi^{(4)}_{25} = \begin{cases} \Pi^{(3)}_{25} & \text{if } D^{(3)}_{25} \leq D^{(3)}_{24} + D^{(3)}_{45} \\ \Pi^{(3)}_{45} & \text{if } \color{red}{D^{(3)}_{25} > D^{(3)}_{24} + D^{(3)}_{45}} \end{cases} = 1$$

$$\Pi^{(4)}_{31} = \begin{cases} \Pi^{(3)}_{31} & \text{if } D^{(3)}_{31} \leq D^{(3)}_{34} + D^{(3)}_{41} \\ \Pi^{(3)}_{41} & \text{if } \color{red}{D^{(3)}_{31} > D^{(3)}_{34} + D^{(3)}_{41}} \end{cases} = 4$$

$$\Pi^{(4)}_{35} = \begin{cases} \Pi^{(3)}_{35} & \text{if } D^{(3)}_{35} \leq D^{(3)}_{34} + D^{(3)}_{45} \\ \Pi^{(3)}_{45} & \text{if } \color{red}{D^{(3)}_{35} > D^{(3)}_{34} + D^{(3)}_{45}} \end{cases} = 1$$

Working Example



$$\Pi^{(3)} = \begin{bmatrix} NIL & 1 & 1 & 2 & 1 \\ NIL & NIL & NIL & 2 & 2 \\ NIL & 3 & NIL & 2 & 2 \\ 4 & 3 & 4 & NIL & 1 \\ NIL & NIL & NIL & 5 & NIL \end{bmatrix}$$

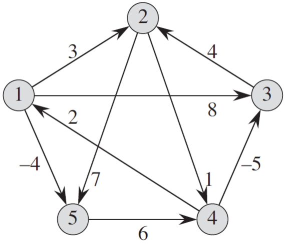
$$\Pi^{(4)} = \begin{bmatrix} NIL & 1 & \mathbf{4} & 2 & 1 \\ \mathbf{4} & NIL & \mathbf{4} & 2 & \mathbf{1} \\ \mathbf{4} & 3 & NIL & 2 & \mathbf{1} \\ 4 & 3 & 4 & NIL & 1 \\ \mathbf{4} & \mathbf{3} & \mathbf{4} & 5 & NIL \end{bmatrix}$$

$$\Pi^{(4)}_{51} = \begin{cases} \Pi^{(3)}_{51} & \text{if } D^{(3)}_{51} \le D^{(3)}_{54} + D^{(3)}_{41} \\ \Pi^{(3)}_{41} & \text{if } D^{(3)}_{51} > D^{(3)}_{54} + D^{(3)}_{41} \end{cases} = 4$$

$$\Pi^{(4)}_{52} = \begin{cases} \Pi^{(3)}_{52} & \text{if } D^{(3)}_{52} \le D^{(3)}_{54} + D^{(3)}_{42} \\ \Pi^{(3)}_{42} & \text{if } D^{(3)}_{52} > D^{(3)}_{54} + D^{(3)}_{42} \end{cases} = 3$$

$$\Pi^{(4)}_{53} = \begin{cases} \Pi^{(3)}_{53} & \text{if } D^{(3)}_{53} \le D^{(3)}_{54} + D^{(3)}_{43} \\ \Pi^{(3)}_{43} & \text{if } D^{(3)}_{53} > D^{(3)}_{54} + D^{(3)}_{43} \end{cases} = 4$$

Working Example



$$D^{(4)} = \begin{bmatrix} 0 & 3 & -1 & 4 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{bmatrix}$$

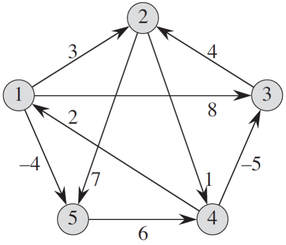
$$D^{(5)} = \begin{bmatrix} 0 & \mathbf{1} & \mathbf{-3} & \mathbf{2} & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{bmatrix}$$

$$D^{(5)}_{12} = \min \begin{cases} D^{(4)}_{12} & = 3 \\ D^{(4)}_{15} + D^{(4)}_{52} & = -4 + 5 \end{cases} = 1$$

$$D^{(5)}_{13} = \min \begin{cases} D^{(4)}_{13} & = -1 \\ D^{(4)}_{15} + D^{(4)}_{53} & = -4 + 1 \end{cases} = -3$$

$$D^{(5)}_{14} = \min \begin{cases} D^{(4)}_{14} & = 4 \\ D^{(4)}_{15} + D^{(4)}_{54} & = -4 + 6 \end{cases} = 2$$

Working Example



$$\Pi^{(4)} = \begin{bmatrix} \text{NIL} & 1 & 4 & 2 & 1 \\ 4 & \text{NIL} & 4 & 2 & 1 \\ 4 & 3 & \text{NIL} & 2 & 1 \\ 4 & 3 & 4 & \text{NIL} & 1 \\ 4 & 3 & 4 & 5 & \text{NIL} \end{bmatrix}$$

$$\Pi^{(5)} = \begin{bmatrix} \text{NIL} & \color{red}{3} & \color{blue}{4} & \color{red}{5} & 1 \\ 4 & \text{NIL} & 4 & 2 & 1 \\ 4 & 3 & \text{NIL} & 2 & 1 \\ 4 & 3 & 4 & \text{NIL} & 1 \\ 4 & 3 & 4 & 5 & \text{NIL} \end{bmatrix}$$

$$\Pi^{(5)}_{12} = \begin{cases} \Pi^{(4)}_{12} & \text{if } D^{(4)}_{12} \leq D^{(4)}_{15} + D^{(4)}_{52} \\ \Pi^{(4)}_{52} & \text{if } \color{red}{D^{(4)}_{12} > D^{(4)}_{15} + D^{(4)}_{52}} \end{cases} = 3$$

$$\Pi^{(5)}_{13} = \begin{cases} \Pi^{(4)}_{13} & \text{if } D^{(4)}_{13} \leq D^{(4)}_{15} + D^{(4)}_{53} \\ \Pi^{(4)}_{53} & \text{if } \color{red}{D^{(4)}_{13} > D^{(4)}_{15} + D^{(4)}_{53}} \end{cases} = 4$$

$$\Pi^{(5)}_{14} = \begin{cases} \Pi^{(4)}_{14} & \text{if } D^{(4)}_{14} \leq D^{(4)}_{15} + D^{(4)}_{54} \\ \Pi^{(4)}_{54} & \text{if } \color{red}{D^{(4)}_{14} > D^{(4)}_{15} + D^{(4)}_{54}} \end{cases} = 5$$

Thank you!