Matrix Chain Multiplication

* Given n matrices
  $A_1 \ A_2$ ——————— A_n
* Multiply in such a way (paranthesize) that minimizes the number of scalar multiplication

Let $A_1 \ A_2 \ A_3 \ A_4$

Explore all ways

$(A_1(A_2(A_3 A_4)))$

$(A_1((A_2 A_3)A_4))$

$((A_1 A_2)(A_3 A_4))$

$((A_1(A_2 A_3))A_4)$

$(((A_1 A_2)A_3)A_4)$

When n matrices

$$\text{Total ways} = \underbrace{\frac{1}{n}\,{}^{2(n-1)}C_{n-1}}_{\text{Catlan no. } n-1^{th}} = \frac{1}{n}\,{}^{2(n-1)}C_{n-1}$$

$n \geq 1$

Total no. of possible BST with $(n-1)$ unique keys

$A = 50 \cdot 20$

$B = 20 \cdot 1$

$C = 1 \cdot 10$

$D = 10 \cdot 100$

A_i dim
$= \underline{\underline{p_{i-1} * p_i}}$

(i) $(A((BC)D))$ $= 20 \cdot 1 \cdot 10 + 20 \cdot 10 \cdot 100 + 50 \cdot 20 \cdot 100$
$= 120\,200$

(ii) $((A(BC))D)$ $= 20 \cdot 1 \cdot 10 + 50 \cdot 20 \cdot 10 + 50 \cdot 10 \cdot 100$
$= 60\,200$

(iii) $((AB)(CD))$ $= 50 \cdot 20 \cdot 1 + 1 \cdot 10 \cdot 100 + 50 \cdot 1 \cdot 100$
$= 7000$

(iv) $(A(B(CD)))$ $= 1 \cdot 10 \cdot 100 + 20 \cdot 1 \cdot 100 + 50 \cdot 20 \cdot 100$

$20 \cdot 1$ $\quad 1 \cdot 100$

$= 1000 + 2000 + 100000$
$= 103000$

(v) $(((AB)C)D)$ $= 50 \cdot 20 \cdot 1 + 50 \cdot 1 \cdot 10 + 50 \cdot 10 \cdot 100$

$50 \cdot 1$ $\quad 1 \cdot 10$

$50 \cdot 10$

$= 1000 + 500 + 50000$
$= 51\,500$

$$\underbrace{(A_i \; A_{i+1} \cdots A_k)}_{X} \underbrace{(A_{k+1} \cdots A_j)}_{Y}$$

1) $A_i \cdots A_k$

$i \leq k < j$

2) $A_{k+1} \cdots A_j$

3) $X \cdot Y$

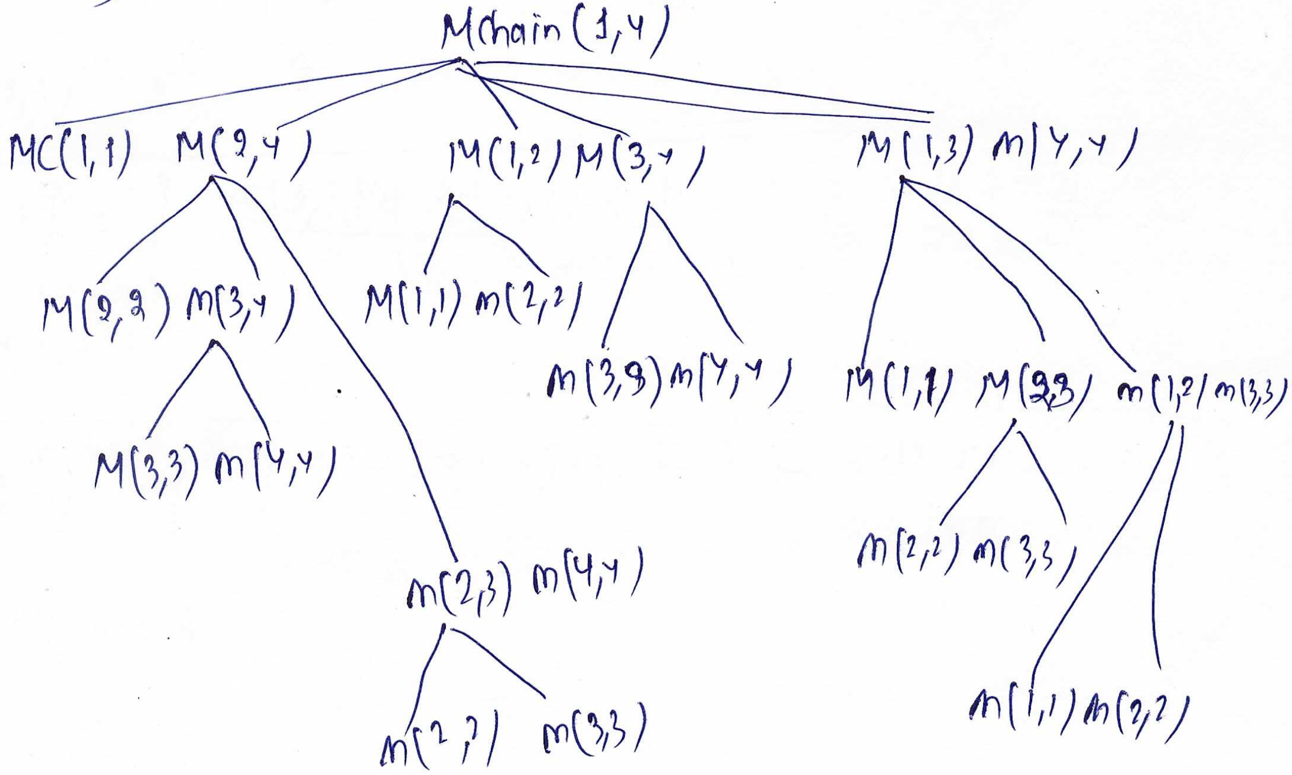
$$\left.\begin{array}{lcl} A_1 & = & p_0 \times p_1 \\ A_2 & = & p_1 \times p_2 \\ \vdots & & \\ A_k & = & p_{k-1} \times p_k \\ \vdots & & \\ A_n & = & p_{n-1} \times p_n \end{array}\right\} \text{Matrix Size}$$

Let $m[i,j]$ = Minimum no. of scalar multiplications needed to compute $(A_i, \cdots A_j)$

$\boxed{\text{Result} \quad M[1,n]}$

Algo

```
MatrixChain ( p, i, j ) {
    if   i = j
         Return 0
    else
         cost ← ∞
         for k = i to j-1
         |   cost ← Min { cost, MatrixChain( p, i, k )
         |          + MatrixChain( p, k+1, j ) }
         |                 + p_{i-1} p_k p_j
         Return cost
}
```



Example

$A_1 = 4 \times 10$
$A_2 = 10 \times 3$
$A_3 = 3 \times 12$
$A_4 = 12 \times 20$
$A_5 = 20 \times 7$

p

0	1	2	3	4	5
4	10	3	12	20	7

$M[1,1] = M[2,2] = M[3,3] = M[4,4] = M[5,5] = 0$

$M[1,2] = \left\{ \underset{K=1}{M(1,1) + M(2,2)} + p_0 \cdot p_1 \cdot p_2 = 4 \cdot 10 \cdot 3 \right.$
$= 120$

$S(1,2) = 1$

$M(2,3) = \left\{ \underset{K=2}{m(2,2) + m(3,3)} + p_1 \cdot p_2 \cdot p_3 = 10 \cdot 3 \cdot 12 = 360 \right.$

$S(2,3) = 2$

$M(3,4) = \left\{ \underset{K=3}{m(3,3) + m(4,4)} + p_2 \cdot p_3 \cdot p_4 = 3 \cdot 12 \cdot 20 \right.$
$= 720$

$S(3,4) = 3$

$M(4,5) = \left\{ \underset{K=4}{M(4,4) + m(5,5)} + p_3 \cdot p_4 \cdot p_5 = 12 \cdot 20 \cdot 7 \right.$
$= 1680$

$S(4,5) = 4$

(7)

$$M[1,3] = \begin{cases} \underbrace{M(1,1)+M(2,3)}_{K=1} + 4\cdot10\cdot12 = 0+360+480 \\ \underbrace{M(1,2)+M(3,3)}_{K=2} + 4\cdot3\cdot12 = 120+0+144 \\ \qquad = 264 \end{cases}$$

Min = 264

$S(1,3) = 2$

$$M[2,4] = \begin{cases} \underbrace{M(2,2)+M(3,4)}_{K=2} + 10\cdot3\cdot20 = 0+720+600 = 1320 \\ \underbrace{M(2,3)+M(4,4)}_{K=3} + 10\cdot12\cdot20 = 360+0+2400 \end{cases}$$

Min = 1320

$S(2,4) = 2$

$$M[3,5] = \begin{cases} \underbrace{M(3,3)+M(4,5)}_{K=3} + 3\cdot12\cdot7 = 0+1680+252 \\ \underbrace{M(3,4)+M(5,5)}_{K=4} + 3\cdot20\cdot7 = 720+0+420 \\ \qquad = 1140 \end{cases}$$

Min = 1140

$S(3,5) = 4$

$$M(1,4) = \begin{cases} \underbrace{M(1,1)+M(2,4)}_{K=1} + 4\cdot10\cdot20 = 0+1320+800 \\ \underbrace{M(1,2)+M(3,4)}_{K=2} + 4\cdot3\cdot20 = \underbrace{120+720+240}_{\text{Min}} \\ \underbrace{M(1,3)+M(4,4)}_{K=3} + 4\cdot12\cdot20 = 264+0+960 \end{cases}$$

$S(1,4) = 2$

$$M(2,5) = \begin{cases} \underbrace{M(2,2) + M(3,5)}_{K=2} + 10 \cdot 3 \cdot 7 = \underline{0 + 1140 + 210} \text{ min} \\ \underbrace{M(2,3) + M(4,5)}_{K=3} + 10 \cdot 12 \cdot 7 = 360 + 1680 + 840 \\ \underbrace{M(2,4) + M(5,5)}_{K=4} + 10 \cdot 20 \cdot 7 = 1320 + 0 + 1400 \end{cases}$$

$S(2,5) = 2$

$$M(1,5) = \begin{cases} \underbrace{M(1,1) + M(2,5)}_{K=1} + 4 \cdot 10 \cdot 7 = 0 + 1350 + 280 \\ \underbrace{M(1,2) + M(3,5)}_{K=2} + 4 \cdot 3 \cdot 7 = \underline{\underline{120 + 1140 + 84}} \text{ min} \\ \underbrace{M(1,3) + M(4,5)}_{K=3} + 4 \cdot 12 \cdot 7 = 264 + 1680 + 336 \\ \underbrace{M(1,4) + M(5,5)}_{K=4} + 4 \cdot 20 \cdot 7 = 1080 + 0 + 560 \end{cases}$$

$S(1,5) = 2$

i \ j	1	2	3	4	5
1	0	120	264	1080	1344
2		0	360	1320	1350
3			0	720	1140
4				0	1680
5					0

i \ j	2	3	4	5
1	1	2	2	2
2		2	3	2
3			3	4
4				4

Min. No of scaller multipliacti = $m(1,5) = 1344$

$A_1 A_2 A_3 A_4 A_5$ What is paranthesis ?????

(i) Check $s(1,5) = 2$ so

$(A_1 A_2)(A_3 A_4 A_5)$

2) Check $s(3,5) = 4$ so

$((A_1 A_2)((A_3 A_4) A_5))$

Final Answer